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First Order Signatures and Knot Concordance

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Abstract

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Invariants of knots coming from twisted signatures have played a central role in the study of knot concordance. Unfortunately, except in the simplest of cases, these signature invariants have proven exceedingly difficult to compute. As a consequence, many knots which presumably can be detected by these invariants are not as well understood as they should be.

We study a family of signature invariants of knots and show that they provide concordance information. Significantly, we provide a tractable means for their computation. Once armed with these tools we use them to study the knot concordance group generated by the twist knots which are of order 2 in the algebraic concordance group. We show that, with only finitely many exceptions, these knots form a linearly independent set in the concordance group.

We go on to study a procedure given by Cochran-Harvey-Leidy which produces infinite rank subgroups of the knot concordance group which, in some sense are extremely subtle and difficult to detect. The construction they give has an inherent ambiguity due to the difficulty of computing certain signature invariants. This ambiguity prevents their construction from yielding an explicit linearly independent set. Using the tools we develop, we make progress in removing this ambiguity from their procedure.

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Chapter 1

Introduction

A knot is an isotopy class of smooth embeddings of the circle, S^1, into the 3-sphere, S^3. A pair of knots, K and J, is called concordant if there is a locally flat embedding of the annulus $S^1 \times [0,1]$ into $S^3 \times [0,1]$ mapping $S^1 \times \{1\}$ to a representative of K in $S^3 \times \{1\}$ and $S^1 \times \{0\}$ to a representative of J in $S^3 \times \{0\}$. A knot is called slice if it is concordant to the standard trivial knot or equivalently if it is the boundary of a locally flat embedding of the 2-ball B^2 into the 4-ball B^4. The set of concordance classes of knots (under the operation of connected sum) forms an abelian group, $\mathcal{C}$, called the knot concordance group.

Recall that an embedding h of an n-manifold F into an m-manifold W is called locally flat if there is an open neighborhood, N, of $h[F]$, the image of h, such that the pair $(N, h[F])$ is locally homeomorphic to $(\mathbb{R}^n \times \mathbb{R}^{m-n}, \mathbb{R}^n \times \{0\})$. For example, smooth embeddings are locally flat.

In the 1960's, Levine [Lev69] defined a surjection from $\mathcal{C}$ to $\mathbb{Z}^\infty \oplus \mathbb{Z}_2^\infty \oplus \mathbb{Z}_4^\infty$ and in doing so found that $\mathcal{C}$ is infinite rank. The kernel of this homomorphism is the group of algebraically slice knots. The quotient of $\mathcal{C}$ by the algebraically slice knots is called the algebraic concordance group and is denoted $\mathcal{AC}$. For twenty years it was not known if this provided a complete description of knot concordance, that is, if

$\mathcal{C} \cong \mathcal{AC}$.

In the 1980's, Casson and Gordon in [CG78], using a family of invariants now called the Casson-Gordon invariants, found that of the algebraically slice twist knots (depicted in Figure 1.1), only the 0 and 2-twist knots are slice. In doing so, they produced the first nonslice, algebraically slice knots. Jiang in [Jia81] used a refinement of these invariants to show that the algebraically slice twist knots are linearly independent in $\mathcal{C}$, so that the kernel of $\mathcal{C} \to \mathcal{AC}$ has infinite rank.

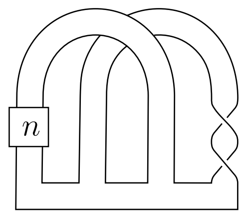

Figure 1.1: The n-twist knot, T_n. Here the n indicates n full twists.

In 2003, [COT03] Cochran, Orr and Teichner gave a filtration of $\mathcal{C}$ by subgroups indexed by half integers, $\cdots \leq \mathcal{F}_{n.5} \leq \mathcal{F}_n \leq \cdots \leq \mathcal{F}_0 \leq \mathcal{C}$. A knot in $\mathcal{F}_n$ (or $\mathcal{F}_{n.5}$) is called (n)-solvable (or $(n.5)$-solvable). They showed that the Casson-Gordon invariants vanish on $\mathcal{F}_{1.5}$. By showing that $\mathcal{F}_2/\mathcal{F}_{2.5}$ is nontrivial they produced the first nonslice knots which the Casson-Gordon invariant obstruction does not detect.

For $n > 2$, the successive quotients, $\mathcal{F}_n/\mathcal{F}_{n.5}$, were shown to be nontrivial by Cochran-Teichner in 2007 [CT07]. They were shown to have infinite rank by Cochran-Harvey-Leidy in 2009 [CHL09]. An infinitely generated 2-torsion subgroup is detected in [CHL11]. Little more is known about these quotients and much remains to be discovered.

From a knot, K, one can obtain a 3-manifold with boundary by considering the complement of an open neighborhood of K in S^3, $E(K)$. The boundary of $E(K)$ is a torus. The <u>meridian</u> of K refers to the isotopy class of simple closed curves on this torus which bound embedded disks in the neighborhood of K which intersect K positively in one point. The <u>longitude</u> of K refers to the isotopy class of simple closed curves on this torus which are isotopic in the neighborhood of K to the original knot, K, and which bound a surface in $E(K)$. Such a surface is called a Seifert surface for K. One can get a closed three-manifold by gluing a solid torus to $E(K)$. If one does so in such a manner as to make the longitude of K bound a disk, one obtains the a closed manifold, $M(K)$, called the zero surgery on K.

An important tool in the study of $(n.5)$-solvability is the von Neumann ρ-invariant. It was defined in the 1980's by Cheeger and Gromov [CG85]. It associates a real number to representations of the fundamental group of a closed 3-manifold. The von Neumann ρ-invariants associated to the zero surgery are powerful obstructions to a knot being highly solvable. For examples of the use of this tool in knot concordance, see [COT03, CT07, CHL10b, Fri05].

In Chapter 3 we define a family of ρ-invariants in terms of metabelian representations which provide information about $\mathcal{C}/\mathcal{F}_{1.5}$. We refer to these as first order signatures of the knot. In Chapter 4 we find a means of approximating first order signatures in terms of ρ-invariants associated to links and homomorphisms to free abelian groups. In Chapter 5 we show that these "abelian ρ-invariants" can be computed in terms of a simple function defined by Cimasoni-Florens in [CF08].

Combining the results of Chapters 3, 4, and 5 we have a computable tool which reveals potentially good information about $\mathcal{C}/\mathcal{F}_{1.5}$. In Chapter 6 we apply this invariant to the study of the twist knots which are of order 2 in $\mathcal{AC}$ and show that with only finitely many exceptions they are linearly independent in $\mathcal{C}/\mathcal{F}_{1.5}$.

In [CHL09], each of the terms, $\mathcal{F}_n/\mathcal{F}_{n.5}$, are shown to contain infinite rank free abelian subgroups. This result is strengthened in [CHL10b] to reveal a kind of primary decomposition of $\mathcal{F}_n/\mathcal{F}_{n.5}$. The construction of [CHL10b] depends on the existence of a class of slice knots called robust doubling operators. In that paper, they exhibit an infinite family of slice knots of which, in some sense, at least half are robust doubling operators, although they cannot verify that any member of their family is robust. In Chapter 7, we eliminate this ambiguity from their construction, finding an explicit infinite set of robust doubling operators.

Similarly, the construction in [CHL09] of linearly independent sets in $\mathcal{F}_n/\mathcal{F}_{n.5}$ requires slice knots with a particular nonvanishing ρ-invariant. We remark in passing that the resulting ambiguity is likewise addressed by the results in Chapter 7.

Chapter 2

Background

2.1 The solvable filtration of the knot concordance group

We begin by recalling the definition, due to Cochran-Orr-Teichner, of the solvable filtration of the knot concordance group. Recall that for a group G, the rational derived series of G is defined recursively by

$$\begin{aligned} G_{\mathbb{Q}}^{(0)} &= G, \\ G_{\mathbb{Q}}^{(n+1)} &= \left\{ g \in G_{\mathbb{Q}}^{(n)} : [g] \text{ is torsion in the abelianization of } G_{\mathbb{Q}}^{(n)} \right\}. \end{aligned}$$

It has the defining property that it is the most quickly descending series whose successive quotients, $G_{\mathbb{Q}}^{(n)}/G_{\mathbb{Q}}^{(n+1)}$, are torsion-free-abelian (abbreviated TFA). A group is poly-torsion-free-abelian or PTFA if $G_{\mathbb{Q}}^{(n)}$ vanishes for some n.

Definition 2.1.1. A knot is called (n)-solvable, denoted $K \in \mathcal{F}_n$, if there exists a smooth 4-manifold W, called an (n)-solution, bounded by the zero surgery on K, $M(K)$, such that:

1. The inclusion induced map $H_1(M(K)) \to H_1(W)$ is an isomorphism.

2. $H_2(W)$ has a basis given by smoothly embedded surfaces with product neighborhoods, $\{L_1, D_1, \ldots, L_r, D_r\}$, all disjoint except that L_i intersects D_i transversely in one point.

3. The images of the inclusion induced maps $\pi_1(L_i) \to \pi_1(W)$ and $\pi_1(D_i) \to \pi_1(W)$ are contained in the n'th term of the rational derived series of $\pi_1(W)$, denoted $\pi_1(W)^{(n)}_{\mathbb{Q}}$.

K is called (n.5)-solvable and W an (n.5)-solution if additionally, $\pi_1(L_i)$ sits in the $(n+1)$'th term of the derived series of $\pi_1(W)$.

If K is slice, then let D be a slice disk bounded by K, and $E(D)$ be the complement of a neighborhood of D in B^4. Notice that for every n, $E(D)$ is an (n)-solution for K. Any obstructions to solvability are obstructions to sliceness.

2.2 Von Neumann ρ-invariants

Some of the most important obstructions to a knot being contained in $\mathcal{F}_{n.5}$ are von Neumann ρ-invariants. These certainly are the the most important tools in this thesis. They are defined in terms of the L^2-signature invariant of 4-manifolds. The definition of the von Neumann ρ-invariant we use in this thesis appears in, for example [CT07, equation 2.10, definition 2.11] and [Har08, section 3].

Definition 2.2.1. Consider an oriented 3-manifold, M, with a homomorphism ϕ from $\pi_1(M)$ to some group Γ. Suppose that M is the oriented boundary of a compact oriented 4-manifold W and $\psi : \pi_1(W) \to \Lambda$ is a homomorphism such that there is a monomorphism $\alpha : \Gamma \to \Lambda$ making the following diagram commute:

$$\begin{tikzcd}
\pi_1(M) \arrow[r, "\phi"] \arrow[d] & \Gamma \arrow[d, hook, "\alpha"] \\
\pi_1(W) \arrow[r, "\psi"] & \Lambda
\end{tikzcd}$$

Then $\rho(M, \phi) := \sigma^{(2)}(W, \psi) - \sigma(W)$ where σ is the regular signature of W and $\sigma^{(2)}$ is the L^2-signature of W twisted by the coefficient system ψ. When the maps, ϕ and ψ are clear from context, we will instead use the notation $\rho(M, \Gamma)$ and $\sigma^{(2)}(W, \Lambda)$.

We will recall some properties of signatures and L^2-signatures of 4-manifolds in Section 2.3

A reason from the point of view of knot concordance to study ρ-invariants is the fact that they provide an obstruction to solvability and so to sliceness.

Theorem (Theorem 4.2 of [COT03])**.** *Let $K \in \mathcal{F}_{n.5}$ be an $(n.5)$-solvable knot and W be an $(n.5)$-solution for K. If $\pi_1(W) \to \Gamma$ is a homomorphism to a group with $\Gamma^{(n+1)}_{\mathbb{Q}} = \{0\}$, then $\rho(M(K), \Gamma) = \sigma^{(2)}(W, \Gamma) - \sigma(W) = 0$.*

Thus, if $\Gamma^{(n+1)}_{\mathbb{Q}} = \{0\}$ and $\phi : \pi_1(M(K)) \to \Gamma$ is a homomorphism which would extend over an $(n.5)$-solution, if one exists, then by computing $\rho(M(K), \Gamma)$ one can hope to get an obstruction to $(n.5)$-solvability and so to sliceness.

One of the easiest invariants which fits this model comes from noticing that if W is a (0.5)-solution for K then by definition, $H_1(M(K)) \xrightarrow{i_*} H_1(W) \cong \mathbb{Z}$ is an isomorphism and the following diagram commutes,

$$\begin{tikzcd}
\pi_1(M(K)) \arrow[r, "\phi_{\text{Abe}}"] \arrow[d] & H_1(M(K)) \cong \mathbb{Z} \arrow[d, "\cong"'] \\
\pi_1(W) \arrow[r, "\phi_{\text{Abe}}"] & H_1(W) \cong \mathbb{Z}
\end{tikzcd}$$

where ϕ_{Abe} denotes the abelianization map. Thus, [COT03, 4.2] applies and $\rho(M(K), \mathbb{Z}) = 0$. Notice that this invariant, $\rho(M(K), \mathbb{Z})$, makes no mention of the (0.5)-solution,

W. Since we will need to make reference to this invariant later, we refer to it as $\rho^0(K) := \rho(M(K), \mathbb{Z})$. In [COT04, Proposition 5.1], Cochran, Orr and Teichner show that $\rho(M(K), \mathbb{Z})$ is equal to the integral of the Tristram-Levine signature function of K.

2.3 Signature invariants

Recall that von Neumann ρ-invariants are defined in terms of the classical and the twisted L^2-signatures of 4-manifolds. We now briefly discuss these invariants.

Let X be an oriented compact 4-dimensional manifold. The Kronecker map $\kappa : H^2(X;\mathbb{C}) \to \mathrm{Hom}(H_2(X;\mathbb{C}), \mathbb{C})$ is a surjective homomorphism. The intersection form $Q : H_2(X;\mathbb{C}) \to \mathrm{Hom}(H_2(X);\mathbb{C})$ is defined by the composition

$$H_2(X;\mathbb{C}) \xrightarrow{i_*} H_2(X, \partial X;\mathbb{C}) \xrightarrow{P.D.} H^2(X;\mathbb{C}) \xrightarrow{\kappa} \mathrm{Hom}(H_2(X;\mathbb{C}), \mathbb{C})$$

where $i_* : H_2(X;\mathbb{C}) \to H_2(X, \partial X;\mathbb{C})$ is induced by inclusion and $P.D. : H_2(X, \partial X;\mathbb{C}) \to H^2(X;\mathbb{C})$ is the Poincaré duality isomorphism. Equivalently, Q can be thought of as a bilinear form $H_2(X;\mathbb{C}) \times H_2(X;\mathbb{C}) \to \mathbb{C}$ by taking $Q(a, b) = (Q(a))(b)$. This form is symmetric. The spectral theorem applies to decompose $H_2(X;\mathbb{C})$ as $H_2^+(X) \oplus H_2^-(X) \oplus H_2^0(X)$ such that Q is positive definite on $H_2^+(X)$, negative definite on $H_2^-(X)$ and zero on $H_2^0(X)$.

The classical signature of X is given by $\sigma(X) := \dim_{\mathbb{C}}(H_2^+(X)) - \dim_{\mathbb{C}}(H_2^-(X))$. For the sake of concreteness, observe that this is the difference between the numbers of positive and negative eigenvalues of Q.

The L^2-signature is defined analogously, but in terms of the twisted L^2-homology of X, which we now recall. Let $\pi_1(X) \to \Gamma$ be a group homomorphism, and let $\widetilde{X}_\Gamma$

be the corresponding cover. The associated chain complex,

$$(C_*(\widetilde{X}_\Gamma), \partial_*) = \left[\dots \overset{\partial_{k+1}}{\to} C_k(\widetilde{X}_\Gamma) \overset{\partial_k}{\to} C_{k-1}(\widetilde{X}_\Gamma) \overset{\partial_{k-1}}{\to} \dots \right]$$

consists of $\mathbb{Z}[\Gamma]$-modules and $\mathbb{Z}[\Gamma]$-module homomorphisms. The Hilbert space

$$l^2(\Gamma) := \left\{ \sum_{g \in \Gamma} a_g g : a_g \in \mathbb{C}, \sum_{g \in \Gamma} |a_g|^2 < \infty \right\}$$

is likewise a $\mathbb{Z}[\Gamma]$-module. Consider the tensored chain complex

$$(C_*^{(2)}(\widetilde{X}_\Gamma), \partial_*^{(2)}) = \left[\dots \overset{\partial_{k+1} \otimes 1}{\to} C_k(\widetilde{X}_\Gamma) \otimes l^2(\Gamma) \overset{\partial_k \otimes 1}{\to} C_{k-1}(\widetilde{X}_\Gamma) \otimes l^2(\Gamma) \overset{\partial_{k-1} \otimes 1}{\to} \dots \right].$$

The L^2-homology of X is defined as $H_k^{(2)}(X; l^2(\Gamma)) = \frac{\ker\left(\partial_k^{(2)}\right)}{\mathrm{clos}\left(\mathrm{im}\left(\partial_k^{(2)}\right)\right)}$, where $\mathrm{clos}(V)$ denotes the closure of $V \subseteq l^2(\Gamma)$. According to [Lüc02] these homology groups are projective as modules over the von Neumann algebra, $\mathcal{N}(\Gamma)$, of bounded Γ-equivariant linear operators on $l^2(\Gamma)$. Similarly to the classical setting there is a Hermitian intersection form [LS03]. $Q^{(2)} : H_2^{(2)}(X; l^2(\Gamma)) \times H_2^{(2)}(X; l^2(\Gamma)) \to \mathbb{C}$. The spectral theorem applies to decompose $H_2^{(2)}$ as $H_2^{(2)+}(X) \oplus H_2^{(2)-}(X) \oplus H_2^{(2)0}(X)$. According to [Lüc02] there is a real valued dimension theory for projective $\mathcal{N}(\Gamma)$-modules, $\dim_{\mathcal{N}(\Gamma)}$: $\{\text{projective modules}\} \to \mathbb{R}$, called the <u>von Neumann dimension</u>. The L^2-signature is defined as

$$\sigma^{(2)}(X, \Gamma) = \dim_{\mathcal{N}(\Gamma)}(H_2^{(2)+}(X)) - \dim_{\mathcal{N}(\Gamma)}(H_2^{(2)-}(X)).$$

Contrary to the classical setting, this does not count eigenvalues, of which there may be infinitely many or none at all. For the definition of L^2-homology, the von Neumann dimension and a good survey on L^2-invariants see [Lüc02]. For the more details regarding the L^2-intersection form and the L^2-signature see [LS03].

The L^2-signature has the following important properties

- (The Index Theorem: see Theorem 0.2 of [LS03]) If X is a closed 4-manifold, and $\pi_1(W) \to \Gamma$ is a homomorphism, then $\sigma^{(2)}(W,\Gamma) = \sigma(W)$.

- (Novikov Additivity: see [COT03, Lemma 5.9 part 3]) If W is the union of two other 4-manifold W_1 and W_2 along some common boundary components, then $\sigma^{(2)}(W;\Lambda) = \sigma^{(2)}(W_1;\Lambda) + \sigma^{(2)}(W_2;\Lambda)$

- $\sigma^{(2)}(-W,\Gamma) = -\sigma^{(2)}(W,\Gamma)$.

- If $\Lambda \hookrightarrow \Gamma$ is a monomorphism, then $\sigma^{(2)}(W,\Gamma) = \sigma^{(2)}(W,\Lambda)$

We now discuss the independence of Definition 2.2.1 from the pair $(W, \psi : \pi_1(W) \to \Lambda)$. Indeed, suppose that both $(W_1, \psi_1 : \pi_1(W_1) \to \Lambda_1)$ and $(W_2, \psi_2 : \pi_1(W_2) \to \Lambda_2)$ satisfy that $\pi_1(W_i) = M$ and there is a monomorphism $\Gamma \hookrightarrow \Lambda_i$. Consider the closed oriented 4-manifold $X = W_1 \underset{M}{\cup} -W_2$ gotten by gluing W_1 and W_2 together along their boundary. Since ψ_1 and ψ_2 agree on $\pi_1(M)$, there is a map from $\pi_1(X)$ to the amalgamated free product $\Lambda = \Lambda_1 \underset{\Gamma}{*} \Lambda_2$, into which both Λ_1 and Λ_2 inject. Thus,

$$\begin{aligned} 0 &= \sigma^{(2)}(W,\Lambda) - \sigma(W) && \text{by the index theorem} \\ &= \sigma^{(2)}(W_1,\Lambda_1) - \sigma(W_1) - \big(\sigma^{(2)}(W_2,\Lambda_2) - \sigma(W_2)\big) && \text{by Novikov additivity.} \end{aligned}$$

So that the candidate definitions for $\rho(M,\Gamma)$ coming from (W_1,Λ_1) and (W_2,Λ_2) agree.

An important tool in this paper for getting information about the L^2-signature of a 4-manifold is a bound in terms of the rank of twisted second homology. When Γ is PTFA and more generally whenever $\mathbb{Q}[\Gamma]$ is an Ore domain, $\mathbb{Q}[\Gamma]$ embeds in its skew field of fractions, $\mathcal{K}(\Gamma)$. For the definition of an Ore domain and the Ore localization used to define $\mathcal{K}(\Gamma)$, see [Ste75, Chapter 2].

The twisted chain complexes of X with coefficients in $\mathbb{Q}[\Gamma]$ and $\mathcal{K}(\Gamma)$ are given by

$$(C_*(X;\mathbb{Q}[\Gamma]),\partial_*) = \left[\dots \xrightarrow{\partial_{k+1}} C_k(\widetilde{X}_\Gamma;\mathbb{Q}) \xrightarrow{\partial_k} C_{k-1}(\widetilde{X}_\Gamma;\mathbb{Q}) \xrightarrow{\partial_{k-1}} \dots\right]$$

and

$$(C_*(X;\mathcal{K}(\Gamma)),\partial_*) :=$$
$$\left[\dots \xrightarrow{\partial_{k+1}\otimes 1} C_k(\widetilde{X}_\Gamma;\mathbb{Q})\otimes\mathcal{K}(\Gamma) \xrightarrow{\partial_k\otimes 1} C_{k-1}(\widetilde{X}_\Gamma;\mathbb{Q})\otimes\mathcal{K}(\Gamma) \xrightarrow{\partial_{k-1}\otimes 1} \dots\right].$$

The homology groups $H_k(X;\mathbb{Q}[\Gamma]) = \frac{\ker(\partial_k)}{\operatorname{im}(\partial_{k+1})}$ and $H_*(X;\mathcal{K}(\Gamma)) = \frac{\ker(\partial_k\otimes 1)}{\operatorname{im}(\partial_{k+1}\otimes 1)}$ are given by the homology of these chain complexes. Being a skew field, all finitely generated modules over $\mathcal{K}(\Gamma)$ are free and have a well defined rank.

The L^2-invariants are controlled by these $\mathcal{K}(\Gamma)$-invariants. In particular,

$$\left|\sigma^{(2)}(X,\phi)\right| \leq \operatorname{rank}_{\mathcal{K}(\Gamma)}\left(\frac{H_2\left(X;\mathcal{K}(\Gamma)\right)}{i_*\left[H_2\left(\partial X;\mathcal{K}(\Gamma)\right)\right]}\right), \tag{2.1}$$

where $i_* : H_2\left(\partial W;\mathcal{K}(\Gamma)\right) \to H_2\left(W;\mathcal{K}(\Gamma)\right)$ is the inclusion induced map. This follows from the monotonicity of von Neumann dimension [Lüc02, Lemma 1.4] and the fact that the von Neumann dimension agrees with $\mathcal{K}(\Gamma)$ rank when $\mathbb{Q}[\Gamma]$ is an Ore Domain [Cha08, Lemma 2.4].)

2.4 The localized Alexander module and the Blanchfield form

In this section we recall a construction which has been of central importance in the study of knot concordance. We will make use of it to build our obstruction to (1.5)-solvability.

We begin with the Alexander module. Let X be a CW-complex with infinite cyclic first homology generated by μ. For example X might be zero surgery on a knot, the complement of a slice disk, or an (n)-solution. Let $\widetilde{X}_{\mathbb{Z}}$ be the abelian cover of X. The rational Alexander module of X is defined to be $A_0(X) := H_1(\widetilde{X}_{\mathbb{Z}};\mathbb{Q})$, regarded as a $\mathbb{Q}[t,t^{-1}]$-module by letting t act by the deck translation corresponding to μ. In the language of twisted coefficients $A_0(X) = H_1(X;\mathbb{Q}[t,t^{-1}])$.

A polynomial $p \in \mathbb{Q}[t,t^{-1}]$, p is called symmetric if $p(t) = t^k p(t^{-1})$ for some $k \in \mathbb{Z}$. For such a p, let $R_p := \left\{\frac{r}{q} \in \mathbb{Q}(t) : (p,q) = 1\right\}$ be the localization of $\mathbb{Q}[t,t^{-1}]$ at the multiplicative set $S_p := \{q \in \mathbb{Q}[t,t^{-1}] : (p,q) = 1\}$. By virtue of being a localization, R_p is flat as a $\mathbb{Q}[t,t^{-1}]$-module. Define the localized Alexander module of X by

$$A_0^p(X) := H_1(X;R_p) = A_0(X) \otimes_{\mathbb{Q}[t,t^{-1}]} R_p.$$

In order to make notation easier, for a knot, K, and its zero surgery, $M(K)$, we will abbreviate $A_0^p(M(K))$ by $A_0^p(K)$.

Notice that $\mathbb{Q}[t,t^{-1}]$, R_p, $\mathbb{Q}(t)$, $\mathbb{Q}(t)/\mathbb{Q}[t,t^{-1}]$ and $\mathbb{Q}(t)/R_p$ are equipped with an involution $f(t) \mapsto \overline{f}(t) = f(t^{-1})$. There is a $\mathbb{Q}$-bilinear form called the localized Blanchfield form $\mathfrak{Bl}^p : A_0^p(K) \times A_0^p(K) \to \mathbb{Q}(t)/R_p$. It is Hermitian in that for $f, g \in R_p$ and $a, b \in A_0^p(K)$ $\mathfrak{Bl}^p(fa, gb) = \overline{f}g\mathfrak{Bl}^p(a,b) = \overline{\mathfrak{Bl}(gb, fa)}$.

In order to define the Blanchfield form, we begin by introducing the Bockstein homomorphism. For a symmetric polynomial, p, the short exact sequence $0 \to R_p \xrightarrow{j} \mathbb{Q}(t) \xrightarrow{r} \mathbb{Q}(t)/R_p \to 0$ induces a long exact sequence of cohomology groups.

$$\cdots \to H^q(X;\mathbb{Q}(t)) \xrightarrow{r^*} H^q(X;\mathbb{Q}(t)/R_p) \xrightarrow{B} H^{q+1}(X;R_p) \xrightarrow{j^*} H^{q+1}(X;\mathbb{Q}(t)) \to \dots \tag{2.2}$$

The map B is called the Bockstein homomorphism. Notice that $\mathrm{im}(B)$ is the torsion submodule of $H^{q+1}(X;R_p)$. Thus, the Bockstein homomorphism can be viewed as an epimorphism from $H^q(X;\mathbb{Q}(t)/R_p)$ onto the torsion submodule $T(H^{q+1}(X;R_p))$.

Let X be a compact n-manifold. Let a be an element of $T(H_q(X, \partial X; R_p))$ and b be an element of $T(H_{n-q-1}(X; R_p))$. Then $P.D.(a) \in T(H_{n-q}(X; R_p))$ is in the image of the Bockstein homomorphism. There is some $a' \in H^{n-q-1}(X, \mathbb{Q}(t)/R_p)$ with $B(a') = P.D.(a)$. Let $\kappa : H^{n-q-1}(X, \mathbb{Q}(t)/R_p) \to \mathrm{Hom}_{R_p}(H_{n-q-1}(X, R_p), \mathbb{Q}(t)/R_p)$ denote the Kronecker homomorphism.

Definition 2.4.1 (The Blanchfield form). For elements a of $T(H_q(X, \partial X; R_p))$ and b of $T(H_{n-q-1}(X, R_p))$, $\mathfrak{Bl}^p(a, b) = \kappa(a')(b)$ where $B(a') = a$.

Proposition 2.4.2. *The Blanchfield form does not depend on the choice of a' and so is a well defined map from $T(H_q(X, \partial X; R_p))$ to* $\mathrm{Hom}(T(H_{n-q-1}(X, R_p)), \mathbb{Q}(t)/R_p)$.

Proof. In order to show that $\mathfrak{Bl}^p(a, b)$ does not depend on the choice of a', suppose that a'_1 and a'_2 satisfy $B(a'_1) = B(a'_2) = P.D.(a)$. Then $B(a'_1 - a'_2) = 0$ so that there is some $A \in H^{n-q-1}(X, \mathbb{Q}(t))$ with $j^*(A) = a'_1 - a'_2$ and $\kappa(a'_1)(b) - \kappa(a'_2)(b) = \kappa(j^*(A))(b)$. The following diagram commutes. j is the projection $\mathbb{Q}(t) \to \mathbb{Q}(t)/R_p$.

$$\begin{tikzcd}
H^{n-q-1}(X, \mathbb{Q}(t)) \arrow[r, "j^*"] \arrow[d, "\kappa"'] & H^{n-q-1}(X, \mathbb{Q}(t)/R_p) \arrow[d, "\kappa"'] \\
\mathrm{Hom}(H_{n-q-1}(X; R_p), \mathbb{Q}(t)) \arrow[r, "j_\#"] & \mathrm{Hom}(H_{n-q-1}(X; R_p), \mathbb{Q}(t)/R_p).
\end{tikzcd}$$

Thusm $\kappa(j^*(A))(b) = j^\#(\kappa(A))(b) = j(\kappa(A)(b))$. Since b is torsion and $\kappa(A)$ is a homomorphism to a torsion-free module, it follows that $\kappa(A)(b) = 0$, so that $\kappa(a'_1)(b) = \kappa(a'_2)(b)$, as required. $\square$

The Poincaré duality map is an anti-homomorphism, rather than a homomorphism. That is, for $f \in R_p$ and $a \in H_q(X, \partial X; R_p)$, $P.D.(fa) = \overline{f} P.D.(a)$. The fact that the Blanchfield form is linear in the second entry, but anti-linear in the first is a consequence of this fact.

Of particular significance is the case that $X = M(K)$ for a knot K, $n = 3$, and $p = 1$. By [Coc04, Proposition 3.10], $A_0^p(K) = H_1(M(K); R_p)$ is torsion, the Bockstein homomorphism is an isomorphism, and the localized Blanchfield form is a Hermitian form on $A_0^p(K)$.

In the case that $p = 0$, $S_p \subseteq \mathbb{Q}[t, t^{-1}]$ consists only of units, $R_p = \mathbb{Q}[t, t^{-1}]$ and $A_0^p(K) = A_0(K)$. Thus, $\mathfrak{Bl}^0$ is a Hermitian form on the unlocalized Alexander module. We rename it as $\mathfrak{Bl} := \mathfrak{Bl}^0$ and call it the unlocalized Blanchfield form.

For a submodule $Q \subseteq A_0^p(X)$, the orthogonal complement of Q is given by $Q^\perp := \{s \in A_0^p(X) : \mathfrak{Bl}^p(q, s) = 0 \text{ for all } q \in Q\}$.

Definition 2.4.3. A submodule, $Q \subseteq A_0^p(K)$, is called isotropic if $Q \subseteq Q^\perp$ and is called Lagrangian if $Q = Q^\perp$.

Suppose that W is a (1)-solution for K. Let Q be the kernel of the inclusion induced map $A_0^p(K) = H_1(M(K); R_p) \to A_0^p(W) = H_1(W; R_p)$. With the remainder of this section we recover the result of [COT03, Theorem 4.4] that Q is Lagrangian.

According to [CHL09, Proposition 5.10], the sequence

$$T(H_2(W, M(K); R_p)) \xrightarrow{\partial_*} T(H_1(M(K); R_p)) \xrightarrow{i_*} T(H_1(W); R_p) \qquad (2.3)$$

is exact. Consider the following diagram whose columns give the Blanchfield forms.

The final map on the left hand side is the restriction map.

$$
\begin{tikzcd}
T(H_2(W, M(K); R_p)) \arrow[r, "\partial_*"] \arrow[d, "P.D."] & T(H_1(M(K); R_p)) \arrow[d, "P.D."] \\
T(H^2(W; R_p)) \arrow[r, "i^*"] & T(H^2(M(K); R_p)) \\
H^1(W; \mathbb{Q}(t)/R_p) \arrow[r, "i^*"] \arrow[u, two heads, "B"] \arrow[d, "\kappa"] & H^1(M(K); \mathbb{Q}(t)/R_p) \arrow[u, "B"', "\cong"] \arrow[d, "\kappa"] \\
\mathrm{Hom}_{R_p}\left(H_1(W; R_p), \mathbb{Q}(t)/R_p\right) \arrow[r, "(i_*)^{\mathrm{dual}}"] \arrow[d] & \mathrm{Hom}_{R_p}\left(H_1(M(K); R_p), \mathbb{Q}(t)/R_p\right) \\
\mathrm{Hom}_{R_p}\left(T(H_1(W; R_p)), \mathbb{Q}(t)/R_p\right) \arrow[ur, "(i_*)^{\mathrm{dual}}"] &
\end{tikzcd}
\tag{2.4}
$$

By the exact sequence (2.3), if u, v are in Q, then there are elements U and V of $T(H_2(W, M(K); R_p))$ with $\partial_* U = u$ and $\partial_* V = v$. Thus,

$$
\begin{aligned}
\mathfrak{Bl}^p(u, v) &= ((\kappa \circ B^{-1} \circ P.D.)(u))(v) \\
&= ((\kappa \circ B^{-1} \circ P.D. \circ \partial_*)(U))(\partial_* V).
\end{aligned}
$$

Using the commutativity of the top square in (2.4)

$$
\mathfrak{Bl}^p(u, v) = ((\kappa \circ B^{-1} \circ i^*)(P.D.(U)))(\partial_* V).
$$

Since the Bockstein homomorphism in the left column is surjective, there is some $T \in H^1(W; \mathbb{Q}(t)/R_p)$ with $B(T) = P.D.(U)$. Making this substitution and using the commutativity of the bottom two squares,

$$\begin{aligned}\mathfrak{Bl}^p(u,v) &= ((\kappa\circ B^{-1}\circ i^*\circ B)(T))(\partial_* V)\\ &= ((\kappa\circ i^*)(T))(\partial_* V)\\ &= ((i_*^{\text{dual}}\circ\kappa)(T))(\partial_* V)\\ &= (\kappa(T))(i_*\circ\partial(V)).\end{aligned}$$

This is zero since $i_*\circ\partial_* = 0$. Thus, for any $u,v\in Q$, $\mathfrak{Bl}^p(u,v)=0$ and $Q\subseteq Q^\perp$

Conversely, if $u\in Q^\perp$, then for all $V\in T(H_2(W,M))$, $\mathfrak{Bl}^p(u,\partial(V)) = \mathfrak{Bl}^p(i_*(u),V)$ vanishes. Since R_p is a PID $\mathfrak{Bl}^p$ is nonsingular by [COT03, Theorem 2.13]. Thus, $i_*(u)=0$ and $u\in Q$.

Thus, we have proven the following proposition.

Proposition 2.4.4. *If K is a (1)-solvable knot and W is a (1)-solution for K, then for any symmetric polynomial, p, $\ker(A_0^p(K)\to A_0^p(W))$ is Lagrangian.*

2.4.1 The localized Blanchfield form in terms of the unlocalized Blanchfield form

It follows from [Lei06, Proposition 3.6] that the localized Blanchfield form can be understood in terms of the unlocalized Blanchfield form. In this subsection we recover the following identity.

Proposition 2.4.5. *Let $\Psi:\mathbb{Q}(t)/\mathbb{Q}[t,t^{-1}]\to\mathbb{Q}(t)/R_p$ be the quotient map. Then for any $a,b\in A_0(K)$, and $u,v\in R_p$,*

$$\mathfrak{Bl}^p(a\otimes u, b\otimes v) = \overline{u}v\Psi(\mathfrak{Bl}(a,b)).$$

Proof. Notice by sesquilinearity,

$$\mathfrak{Bl}^p(a \otimes u, b \otimes v) = \overline{u}v\mathfrak{Bl}^p(a \otimes 1, b \otimes 1).$$

It suffices to show that $\mathfrak{Bl}^p(a\otimes 1, b\otimes 1) = \Psi(\mathfrak{Bl}(a, b))$. Let $\psi : \mathbb{Q}[t, t^{-1}] \hookrightarrow R_p$ be the inclusion map and consider the following commutative diagram:

$$\begin{tikzcd}
H_1(M(K); \mathbb{Q}[t, t^{-1}]) \arrow[r, "\psi_*"] \arrow[d, "P.D."] & H_1(M(K); R_p) \arrow[d, "P.D."] \\
H^2(M(K); \mathbb{Q}[t, t^{-1}]) \arrow[r, "\psi^*"] & H^2(M(K); R_p) \\
H^1(M(K); \mathbb{Q}(t)/\mathbb{Q}[t, t^{-1}]) \arrow[u, "B"', "\cong"] \arrow[r, "\Psi^*"] \arrow[d, "\kappa"] & H^1(M(K); \mathbb{Q}(t)/R_p) \arrow[u, "B"', "\cong"] \arrow[d, "\kappa"] \\
\mathrm{Hom}(H_1(M(K); \mathbb{Q}[t, t^{-1}])), \mathbb{Q}(t)/\mathbb{Q}[t, t^{-1}]) \arrow[d, "\Psi_\#"] & \mathrm{Hom}(H_1(M(K); R_p), \mathbb{Q}(t)/R_p) \arrow[dl, "\psi_*^{\mathrm{dual}}"] \\
\mathrm{Hom}(H_1(M(K); \mathbb{Q}[t, t^{-1}]), \mathbb{Q}(t)/R_p) &
\end{tikzcd}$$

where $\Psi_\#$ is defined by $(\Psi_\#(\phi))(x) = \Psi(\phi(x))$ and ψ_*^{dual} is defined by $\left(\psi_*^{\mathrm{dual}}(\phi)\right)(x) = \phi(\psi_*(x))$. Additionally, the map

$$\psi_* : H_1(M(K); \mathbb{Q}[t, t^{-1}]) \to H_1(M(K); R_p) = H_1(M(K); \mathbb{Q}[t, t^{-1}]) \otimes R_p$$

is given by $a \mapsto a \otimes 1$.

Notice that the top three maps in the left and right columns give the unlocalized

Blanchfield form and the localized Blanchfield form respectively. Thus,

$$\begin{aligned}\mathfrak{Bl}^p(a\otimes 1, b\otimes 1) &= ((\kappa\circ B^{-1}\circ P.D.)(\psi_*(a)))(\psi_*(b))\\ &= ((\psi_*^{\text{dual}}\kappa\circ B^{-1}\circ P.D.\circ\psi_*)(a))(b)\end{aligned}$$

which by commutativity is

$$\begin{aligned}\mathfrak{Bl}^p(a\otimes 1, b\otimes 1) &= ((\Psi_\#\circ\kappa\circ B^{-1}\circ P.D.)(a))(b)\\ &= \Psi\left(((\kappa\circ B^{-1}\circ P.D.)(a))(b)\right)\\ &= \Psi(\mathfrak{Bl}(a,b))\end{aligned}$$

as desired. $\square$

Chapter 3

Localized first order signatures

3.1 Defining the invariant

Let X be a CW complex with infinite cyclic first homology and $\widetilde{X}_{\mathbb{Z}}$ be its universal abelian cover. The image of the monomorphism induced by the covering map $\pi_1(\widetilde{X}_{\mathbb{Z}}) \hookrightarrow \pi_1(X)$ is the commutator subgroup $\pi_1(X)^{(1)} = [\pi_1(X), \pi_1(X)]$. Let p be a symmetric polynomial. Consider the following composition:

$$f_p : \pi_1(X)^{(1)} \cong \pi_1(\widetilde{X}) \stackrel{\phi_{\mathrm{Abe}}}{\to} H_1(\widetilde{X};\mathbb{Z}) \to A_0(X) \stackrel{Id \otimes R_p}{\to} A_0^p(X). \tag{3.1}$$

Let $\pi_1(X)^{(2)}_p$ be the kernel of f_p. This subgroup is normal in $\pi_1(X)$ and so it makes sense to discuss the quotient $\frac{\pi_1(X)}{\pi_1(X)^{(2)}_p}$. Notice that if the polynomial p is relatively prime to the characteristic polynomial of $A_0(X)$ then $Id \otimes R_p$ in (3.1) is injective and $\pi_1(X)^{(2)}_p = \pi_1(X)^{(2)}_{\mathbb{Q}}$. Furthermore $\frac{\pi_1(X)}{\pi_1(X)^{(2)}_p}$ fits into the following short exact sequence:

$$0 \to \frac{\pi_1(X)^{(1)}}{\pi_1(X)^{(2)}_p} \to \frac{\pi_1(X)}{\pi_1(X)^{(2)}_p} \to \frac{\pi_1(X)}{\pi_1(X)^{(1)}} \to 0.$$

The first term, $\frac{\pi_1(X)^{(1)}}{\pi_1(X)^{(2)}_p}$, injects into the $\mathbb{Q}$-vector space, $A_0^p(X)$, and so is torsion-free-abelian. The final term, $\frac{\pi_1(X)}{\pi_1(X)^{(1)}} \cong H_1(X)$, is isomorphic to $\mathbb{Z}$ by assumption. Thus, $\left(\frac{\pi_1(X)}{\pi_1(X)_p^{(2)}}\right)_{\mathbb{Q}}^{(2)} = 0$ and $\frac{\pi_1(X)}{\pi_1(X)_p^{(2)}}$ is PTFA.

Consider the case that $X = W$ is a (1.5)-solution for K. By [COT03, Theorem 4.2],

$$\rho\left(M(K), \frac{\pi_1(W)}{\pi_1(W)_p^{(2)}}\right) = 0. \tag{3.2}$$

We wish to replace the quotient, $\frac{\pi_1(W)}{\pi_1(W)_p^{(2)}}$, in equation (3.2) with a group which does not depend on the (1.5)-solution, W. If we can do this, we will have a ρ-invariant which obstructs (1.5)-solvability and hence sliceness.

By [COT03, Proposition 5.13] if M is a 3-manifold, $\phi : \pi_1(M) \to \Lambda$ is a homomorphism and $f : \Lambda \hookrightarrow \Gamma$ is an monomorphism, then $\rho(M, \Gamma) = \rho(M, \Lambda)$. In particular, taking $G = \ker\left(\pi_1(M(K)) \to \pi_1(W) \to \frac{\pi_1(W)}{\pi_1(W)_p^{(2)}}\right)$ then (3.2) reduces to

$$\rho\left(M(K), \frac{\pi_1(M(K))}{G}\right) = \rho\left(M(K), \frac{\pi_1(W)}{\pi_1(W)_p^{(2)}}\right) = 0. \tag{3.3}$$

Thus, a good next step is to describe this subgroup G. We begin by examining the following commutative diagram whose rows are exact sequences.

$$\begin{tikzcd}
0 \arrow[r] & \frac{\pi_1(M(K))^{(1)}}{\pi_1(M(K))_p^{(2)}} \arrow[r] \arrow[d, "\alpha"'] & \frac{\pi_1(M(K))}{\pi_1(M(K))_p^{(2)}} \arrow[r] \arrow[d] & \frac{\pi_1(M(K))}{\pi_1(M(K))^{(1)}} \arrow[r] \arrow[d, "\cong"'] & 0 \\
0 \arrow[r] & \frac{\pi_1(W)^{(1)}}{\pi_1(W)_p^{(2)}} \arrow[r] & \frac{\pi_1(W)}{\pi_1(W)_p^{(2)}} \arrow[r] & \frac{\pi_1(W)}{\pi_1(W)^{(1)}} \arrow[r] & 0
\end{tikzcd}$$

The rightmost vertical map, $\frac{\pi_1(M(K))}{\pi_1(M(K))^{(1)}} \cong H_1(M(K)) \to \frac{\pi_1(W)}{\pi_1(W)^{(1)}} \cong H_1(W)$ is an isomorphism. If the left-most map, α, were a monomorphism, then the five-lemma would apply to show that the central map is also a monomorphism and we would conclude that $\rho\left(M(K), \frac{\pi_1(M(K))}{\pi_1(M(K))_p^{(2)}}\right) = \rho\left(M(K), \frac{\pi_1(W)}{\pi_1(W)_p^{(2)}}\right)$. Unfortunately this is not the case. However, the kernel of α does have some structure. The map α extends to an R_p-module homomorphism $\overline{\alpha}$ between localized Alexander modules.

$$\begin{tikzcd}
\dfrac{\pi_1(M(K))^{(1)}}{\pi_1(M(K))^{(2)}_p} \arrow[r, hook] \arrow[d, "\alpha"] & A_0^p(K) \arrow[d, "\overline{\alpha}"] \\
\dfrac{\pi_1(W)^{(1)}}{\pi_1(W)^{(2)}_p} \arrow[r, hook] & A_0^p(W)
\end{tikzcd}$$

The kernel of $\overline{\alpha}$ is a submodule of $A_0^p(M(K))$. With this construction in mind, we let Q be a submodule of $A_0^p(K)$ and consider the following composition:

$$f_{p,Q} : \pi_1(M(K))^{(1)} \xrightarrow{f_p} A_0^p(K) \to \frac{A_0^p(K)}{Q}.$$

Let $\pi_1(M(K))^{(2)}_{p,Q}$ be the kernel of $f_{p,Q}$. If W is a (1.5)-solution for K and Q is the kernel of the map $A_0^p(K) \to A_0^p(W)$, then the following diagram commutes and has exact rows.

$$\begin{tikzcd}
0 \arrow[r] & \dfrac{\pi_1(M(K))^{(1)}}{\pi_1(M(K))^{(2)}_{p,Q}} \arrow[r] \arrow[d, hook, "\alpha"] & \dfrac{\pi_1(M(K))}{\pi_1(M(K))^{(2)}_{p,Q}} \arrow[r] \arrow[d] & \dfrac{\pi_1(M(K))}{\pi_1(M(K))^{(1)}} \arrow[r] \arrow[d, "\cong"'] & 0 \\
0 \arrow[r] & \dfrac{\pi_1(W)^{(1)}}{\pi_1(W)^{(2)}_p} \arrow[r] & \dfrac{\pi_1(W)}{\pi_1(W)^{(2)}_p} \arrow[r] & \dfrac{\pi_1(W)}{\pi_1(W)^{(1)}} \arrow[r] & 0
\end{tikzcd} \tag{3.4}$$

By the five-lemma, $\frac{\pi_1(M(K))}{\pi_1(M(K))^{(2)}_{p,Q}} \hookrightarrow \frac{\pi_1(W)}{\pi_1(W)^{(2)}_p}$ is a monomorphism and we obtain the following obstruction.

Proposition 3.1.1. *Let $p(t)$ be a symmetric polynomial and K be a (1.5)-solvable knot. Let W be a (1.5)-solution for K and $Q = \ker\left(A_0^p(K) \to A_0^p(W)\right)$.*

Then

$$\rho\left(M(K), \frac{\pi_1(M(K))}{\pi_1(M(K))^{(2)}_{p,Q}}\right) = 0$$

We make the following definition:

Definition 3.1.2 (localized first order signatures)*.* For a knot K, a symmetric polynomial $p(t)$ and a submodule $Q \subseteq A_0^p(K)$,

$$\rho^1_{p(t),Q}(K) := \rho\left(M(K), \frac{\pi_1(M(K))}{\pi_1(M(K))^{(2)}_{p,Q}}\right).$$

In the case that $Q = \{0\}$ is the trivial submodule or $p(t) = 0$, we drop them from the notation:

$$\rho^1(K) := \rho^1_{0,\{0\}}(K), \rho^1_p(K) := \rho^1_{p,\{0\}}(K), \rho^1_Q(K) := \rho^1_{0,Q}(K)$$

Notice that Proposition 2.4.4 reveals that the kernel, Q, of the inclusion induced map $A_0(K) \to A_0(W)$ is Lagrangian and so we obtain the following corollary.

Corollary 3.1.3. *Let $p(t)$ be a symmetric polynomial, and K be a (1.5)-solvable knot. Then there exists a Lagrangian submodule $Q \subseteq A_0^p(K)$ such that*

$$\rho^1_{p,Q}(K) = 0.$$

3.2 Algebraic properties of localized first order signatures and the main obstruction to linear dependence

In this section we prove an important algebraic property of first order signatures, namely that they are additive under connected sum. We begin by building a cobordism between $M(K_1\#K_2)$ and $M(K_1) \sqcup M(K_2)$ which we use to relate their ρ-invariants.

Construction 3.2.1. Let K_1 and K_2 be knots. For $i = 1, 2$ let μ_i be a curve in $M(K_i)$ isotopic to the meridian of K_i. Let $W(K_1, K_2)$ be given by starting with $M(K_1) \times [0,1] \sqcup M(K_2) \times [0,1]$ and adding a copy of $S^1 \times B^2 \times [1,2]$. Glue $S^1 \times B^2 \times \{i\}$ to a neighborhood, N_i, of $\mu_i \times \{1\}$ so that the zero-framed pushoff of μ_i is identified with the curve $S^1 \times \{x\} \times \{ i\}$ for some $x \in \partial B^2$. The boundary of $W(K_1, K_2)$ is given by $M(K_1\#K_2) \sqcup -M(K_1) \sqcup -M(K_2)$.

Let V denote the copy of $S^1 \times B^2 \times [0,1]$ added to $M(K_1) \times [0,1] \sqcup M(K_2) \times [0,1]$ in Construction 3.2.1. Let v denote the curve $S^1 \times \{x\} \times \{ 1\} \subseteq V$ which generates $\pi_1(V)$.

Lemma 3.2.2. *The inclusion induced maps*

$$\begin{tikzcd}
H_1(M(K_1)) \arrow[dr] & \\
H_1(M(K_2)) \arrow[r] & H_1(W(K_1,K_2)) \\
H_1(M(K_1\#K_2)) \arrow[ur] &
\end{tikzcd}$$

are all isomorphisms.

Proof. Consider the Mayer-Vietoris long exact sequence associated to the decomposition of $W(K_1, K_2)$ into $M(K_1) \sqcup M(K_2)$ together with V glued together along $N_1 \sqcup N_2$.

$$\cdots \to H_1(N_1 \sqcup N_2) \to H_1(M(K_1) \sqcup M(K_2)) \oplus H_1(V) \to H_1(W(K_1, K_2)) \to 0.$$

Notice that $H_1(N_i)$ and $N_1(M(K_i))$ are each freely generated by μ_i and that $H_1(V)$

is freely generated by v. Making these substitutions,

$$\cdots \to \langle \mu_1, \mu_2 \rangle \underset{\mu_i \mapsto \mu_1 - v}{\longrightarrow} \langle \mu_1, \mu_2, v \rangle \to H_1(W(K_1, K_2)) \to 0.$$

So $H_1(W(K_1, K_2))$ has the presentation

$$H_1(W(K_1, K_2)) = \langle \mu_1, \mu_2, v | \mu_1 = v, \mu_2 = v \rangle$$

and is freely generated by either of μ_1 or μ_2.

Finally, the meridian of $K_1 \# K_2$ is freely isotopic in $W(K_1, K_2)$ to each of μ_1 and μ_2. This completes the proof. $\square$

One consequence of Lemma 3.2.2 is that the inclusion from any of the boundary components, $M(K_1)$, $M(K_2)$ and $-M(K_1 \# K_2)$, into $W(K_1, K_2)$ induces a homomorphism of Alexander modules. We see that $A_0^p(M(K_1 \# K_2)) \to A_0^p(W(K_1, K_2))$ and $A_0^p(K_1) \oplus A_0^p(K_2) \to A_0^p(W(K_1, K_2))$ are isomorphisms.

Lemma 3.2.3. *Let $p(t)$ be a symmetric polynomial. The inclusion induced maps*

$$A_0^p(K_1 \# K_2) \longrightarrow A_0^p(W(K_1, K_2)) \longleftarrow A_0(K_1) \oplus A_0^p(K_2)$$

are isomorphisms.

Proof. Consider the Mayer Vietoris sequence with coefficients in R_p.

$$\begin{gathered} \cdots \to H_k(N_1 \sqcup N_2; R_p) \to H_k(M(K_1); R_p) \oplus H_k(M(K_2); R_p) \oplus H_1(V; R_p) \to \\ H_k(W(K_1, K_2); R_p) \to H_{k-1}(N_1 \sqcup N_2; R_p) \to \dots. \end{gathered} \tag{3.5}$$

The infinite cyclic cover of V is homeomorphic to $\mathbb{R} \times B^2 \times [1, 2]$ which is contractible. Thus, $H_k(V; R_p) = H_k(\widetilde{V_\mathbb{Z}}) \otimes_{\mathbb{Q}[t,t^{-1}]} R_p = 0$ for $k > 0$. Similarly, $H_k(N_1 \sqcup N_2; R_p) =$

$H_k(N_1;R_p) \oplus H_k(N_2;R_p) = 0$. This observation together with the long exact sequence (3.5) implies that $H_k(M(K_1);R_p) \oplus H_k(M(K_2);R_p) \to H_k(W(K_1,K_2);R_p)$ is an isomorphism. Taking $k = 1$ gives the first isomorphism in Lemma 3.2.3.

We now show $A_0^p(K_1\#K_2) \to A_0(W(K_1,K_2))$ is an isomorphism. The fact that $H_k(M(K_1);R_p) \oplus H_k(M(K_2);R_p) \to H_k(W(K_1,K_2);R_p)$ is an isomorphism for all k implies that $H_k(W(K_1,K_2), M(K_1) \sqcup M(K_2);R_p) = 0$ for all k. Poincare duality and the universal coefficient theorem then assert that $H_k(W(K_1,K_2), M(K_1\#K_2);R_p) = 0$ for all k. Finally, the long exact sequence of the pair, $(W(K_1,K_2), M(K_1\#K_2))$ reveals that $H_k(M(K_1\#K_2);R_p) \to H_k(W(K_1,K_2);R_p)$ is an isomorphism. $\square$

Let f be the isomorphism given by the composition

$$A_0^p(K_1) \oplus A_0^p(K_2) \cong A_0(W(K_1,K_2)) \cong A_0(K_1\#K_2).$$

Lemma 3.2.4. *The homomorphism, f, preserves Blanchfield forms, that is for $(a_1 \oplus a_2), (b_1 \oplus b_2) \in A_0^p(K_1) \oplus A_0^p(K_2)$,*

$$\mathfrak{Bl}^p(f(a_1 \oplus a_2), f(b_1 \oplus b_2)) = \mathfrak{Bl}^p(a_1, b_2) + \mathfrak{Bl}^p(a_2, b_2).$$

Proof. Let $\mathfrak{Bl}^p_{\partial W} : H_1(\partial W(K_1,K_2);R_p) \times H_1(\partial W(K_1,K_2);R_p) \to \mathbb{Q}(t)/R_p$ be the Blanchfield form on ∂W. By the same analysis as surrounds equation (2.4), the kernel of the inclusion induced map $j_* : H_1(\partial W(K_1,K_2);R_p) \to H_1(W(K_1,K_2);R_p)$ is isotropic with respect to this form. For $i = 1,2$ let $c_i \in A_0^p(K_i)$. Notice that $(c_1 + c_2 - f(c_1 \oplus c_2)) \in A_0^p(K_1) \oplus A_0^p(K_2) \oplus A_0^p(K_1\#K_2)$ is in the kernel of j. Thus,

$$0 = \mathfrak{Bl}^p_{\partial W(K_1,K_2)}(a_1 + a_2 - f(a_1 \oplus a_2), b_1 + b_2 - f(b_1 \oplus b_2)).$$

We expand and take advantage of the fact that if x and y are elements of $H_1(\partial W;R_p)$

carried by different components of ∂W, then $\mathfrak{Bl}^p_{\partial W}(x,y) = 0$.

$$0 = \mathfrak{Bl}^p_{\partial W(K_1,K_2)}(a_1,b_1) + \mathfrak{Bl}^p_{\partial W(K_1,K_2)}(a_2,b_2) + \mathfrak{Bl}^p_{\partial W(K_1,K_2)}(f(a_1 \oplus a_2), f(b_1 \oplus b_2)).$$

Finally, since $\partial W(K_1,K_2) = M(K_1\#K_2) \sqcup -M(K_1) \sqcup -M(K_2)$, this reduces to $0 = -\mathfrak{Bl}^p(a_1,b_1) - \mathfrak{Bl}^p(a_2,b_2) + \mathfrak{Bl}^p(f(a_1 \oplus a_2), f(b_1 \oplus b_2))$, proving the lemma. $\square$

The map f provides an identification of $A_0^p(K_1\#K_2)$ with $A_0^p(K_1) \oplus A_0^p(K_2)$. Lemma 3.2.4 shows that it respects the Blanchfield form. We will now treat $A_0^p(K_1)$ and $A_0^p(K_2)$ as submodules of $A_0^p(K_1\#K_2)$.

Corollary 3.2.5. *Let Q be a submodule of $A_0^p(K_1\#K_2) = A_0^p(K_1) \oplus A_0^p(K_2)$. For $i = 1,2$, let $Q_i = Q \cap A_0^p(K_i)$ and $j_*^+ : A_0^p(K_1\#K_2) \to A_0^p(W(K_1,K_2)$ be the inclusion induced map.*

1. $\frac{A_0^p(K_1\#K_2)}{Q} \to \frac{A_0(W(K_1,K_2))}{j_*^+[Q]}$ *is an isomorphism.*

2. *For $i = 1,2$,* $\frac{A_0^p(K_i)}{Q_i} \to \frac{A_0(W(K_1,K_2))}{j_*^+[Q]}$ *is an monomorphism.*

3. *If Q is isotropic, then so are Q_1 and Q_2.*

By analysis identical to that surrounding equation (3.4), the map

$$\frac{\pi_1(M(K_1\#K_2))}{\pi_1(M(K_1\#K_2))^{(2)}_{p,Q}} \xrightarrow{\cong} \frac{\pi_1(W(K_1,K_2))}{\pi_1(W(K_1,K_2))^{(2)}_{p,j_*^+[Q]}}$$

is an isomorphism and for $i = 1,2$ the map

$$\frac{\pi_1(M(K_i))}{\pi_1(M(K_i))^{(2)}_{p,Q_i}} \hookrightarrow \frac{\pi_1(W(K_1,K_2))}{\pi_1(W(K_1,K_2))^{(2)}_{p,j_*^+[Q]}}$$

is a monomorphism. Thus, by Definition 2.2.1, it follows that

$$\rho^1_{p,Q}(K_1 \# K_2) - \rho^1_{p,Q_1}(K_1) - \rho^1_{p,Q_2}(K_2) =$$
$$\sigma^2\left(W(K_1,K_2), \frac{\pi_1(W(K_1,K_2))}{\pi_1(W(K_1,K_2))^{(2)}_{p,i_*^+[Q]}}\right) - \sigma(W(K_1,K_2)).$$

Finally, we prove that the signature defect on the right hand side of this equality is zero.

Proposition 3.2.6. *1.* $\sigma(W(K_1,K_2)) = 0$

2. For a nontrivial homomorphism to a PTFA group, $\phi : \pi_1(W(K_1,K_2)) \to \Gamma$, $\sigma^{(2)}(W(K_1,K_2),\Gamma) = 0$.

Proof. Recall that N_1 and N_2 are the tubular neighborhoods of the neighborhoods of the meridians μ_1 and μ_2 and that V is the copy of $S^1 \times B^2 \times [0,1]$ glued between them to get $W(K_1,K_2)$. Consider the untwisted Mayer Vietoris sequence

$$H_2(M(K_1) \sqcup M(K_2)) \oplus H_2(V) \xrightarrow{i_*} H_2(W(K_1,K_2)) \xrightarrow{\partial_*} H_1(N_1 \sqcup N_2)$$
$$\hookrightarrow H_1(M(K_1) \sqcup M(K_2)) \oplus H_1(V).$$

As was shown in the proof of Lemma 3.2.2, the final map is injective, so ∂_* is the zero map and i_* is surjective. Since $V \sim S^1$ is a homotopy one-complex, $H_2(V) = 0$. We conclude that $H_2(M(K_1) \sqcup M(K_2)) \to H_2(W(K_1,K_2))$ is surjective. Since $M(K_1)$ and $M(K_2)$ are components of the boundary of W, $H_2(\partial W) \to H_2(W)$ is surjective and $\sigma(W(K_1,K_2)) = 0$.

Similarly, consider the Mayer-Vietoris long exact sequence with coefficients in $\mathbb{Q}[\Lambda]$

$$H_2(M(K_1) \sqcup H_2(M(K_2)); \mathbb{Q}[\Lambda]) \oplus H_2(V; \mathbb{Q}[\Lambda]) \to$$
$$H_2(W(K_1,K_2); \mathbb{Q}[\Lambda]) \to H_1(N_1 \sqcup N_2; \mathbb{Q}[\Lambda]).$$

The manifolds $M(K_1), M(K_2), V, N_1,$ and N_2 are all (homotopy equivalent to) 3-

manifolds with infinite cyclic first homology. Each of them has fundamental group normally generated by μ_1 or μ_2, which are isotopic in Y and normally generate $\pi_1(W(K_1, K_2))$. Since ϕ is nontrivial, it follows that $\phi(\mu_1) \neq 1$ and the homomorphism to Γ is nontrivial on each of $M(K_1), M(K_2), V, N_1$, and N_2. Corollary 3.12 of [Coc04] concludes that the homology of each of these spaces with coefficients in $\mathbb{Q}[\Lambda]$ is torsion. It follows then that $H_2(W(K_1, K_2); \mathbb{Q}[\Lambda])$ is torsion and by the inequality (2.1), $\sigma^{(2)}(W(K_1, K_2), \Lambda) = 0$. $\square$

Putting this all together we get the following proposition.

Proposition 3.2.7. *For a symmetric polynomial p and knots K_1 and K_2, let $K = K_1 \# K_2$ and Q be a submodule of $A_0^p(K) = A_0^p(K) \oplus A_0^p(J)$. For $i = 1, 2$, $Q_i = Q \cap A_0^p(K_1)$. Then $\rho^1_{p,Q}(K_1 \# K_2) = \rho^1_{p,Q_1}(K_1) + \rho^1_{p,Q_2}(K_2)$. If Q is isotropic, then so are Q_1 and Q_2.*

Theorem 3.2.8. *For knots $K_1 \dots K_n$, if $K = \mathop{\#}\limits_{i=1}^{n} K_i$ is (1.5)-solvable, then for every symmetric polynomial p, there exist isotropic submodules $Q_1, \dots Q_n$, $Q_i \subseteq A_0^p(K_i)$ with $\sum_{i=1}^n \rho^1_{p,Q_i}(K_i) = 0$*

Proof. By Proposition 3.1.1, it follows that there is an isotropic submodule $Q \subseteq A_0^p(K)$ with $\rho^1_{p,Q}(K) = 0$. By repeatedly applying Lemma 3.2.7 we see that there are isotropic submodules $Q_i = A_0^p(K_i) \cap Q$ with $\sum\limits_{i=1}^{n} \rho^1_{p,Q_i}(K_i) = \rho^1_{p,Q}(K) = 0$, as was claimed. $\square$

Remark 3.2.9. Even if Q were Lagrangian in the hypothesis of Proposition 3.2.7, it would not follow that Q_1 and Q_2 are Lagrangian. Thus, in Theorem 3.2.8 we cannot replace the word isotropic with the word Lagrangian.

3.3 Families of knots with coprime Alexander polynomial

In this section we restrict Theorem 3.2.8 to the situation that there is a linear dependence amongst a family of knots with pairwise coprime Alexander polynomials. The following reduction of first order signatures is of central importance.

Proposition 3.3.1. *Let Δ be the Alexander polynomial of a knot K.*

1. *If p is a symmetric polynomial relatively prime to Δ, then $A_0^p(K) = 0$, the only isotopic submodule is the trivial submodule, and $\rho_p^1(K) = \rho^0(K)$.*
2. *If $\Delta = p$ then the map $A_0(K) \to A_0^p(K)$ is a monomorphism. A submodule $Q^p \subseteq A_0^p(K)$ is isotropic if and only if $Q^p = Q \otimes R_p$ with Q isotropic. Furthermore, $\rho^1_{p,Q^p}(K) = \rho^1_Q(K)$.*

Proof. In order to see the first claim notice that since Δ annihilates $A_0(K)$ and is invertible in R_p, $A_0^p(K) = A_0(K) \otimes R_p$ is the trivial module. In order to compute ρ_p^1, consider the map $f_p : \pi_1(M(K))^{(1)} \to A_0^p(K) = 0$ in terms of which ρ_p^1 is defined. Since its codomain is trivial, its kernel, which by definition is $\pi_1(M(K))_p^{(2)}$, is equal to $\pi_1(M(K))^{(1)}$. Thus,

$$\rho_p^1(K) = \rho\left(M(K), \frac{\pi_1(M(K))}{\pi_1(M(K))_p^{(2)}}\right) = \rho\left(M(K), \frac{\pi_1(M(K))}{\pi_1(M(K))^{(1)}}\right) = \rho^0(K),$$

as claimed.

If $p = \Delta$, then the kernel of the localization map $A_0(K) \to A_0^p(K)$ is the submodule of $A_0(K)$ consisting of elements annihilated by polynomials coprime to Δ. Since Δ is the characteristic polynomial of $A_0(K)$, the only such element is 0 and $A_0(K) \hookrightarrow A_0^p(K)$ is injective.

Any elements $a_p, b_p \in A_0^p(K)$ are realized as $a_p = a \otimes \frac{1}{r}, b_p = b \otimes \frac{1}{s}$, with $a, b \in A_0$, and $\frac{1}{r}, \frac{1}{s} \in R_p$. By Proposition 2.4.5 the localized Blanchfield can be expressed in terms of the unlocalized form: $\mathfrak{Bl}^p(a_p, b_p) = \frac{1}{\overline{r}s}\Psi(\mathfrak{Bl}(a,b))$ where $\Psi : \mathbb{Q}(t)/\mathbb{Q}[t,t^{-1}] \to \mathbb{Q}(t)/R_p$ is the natural quotient map. Notice that for any $b \in A_0(K)$, $\Delta(t)b = 0$, so that $\mathfrak{Bl}(a, \Delta(t)b) = \Delta(t)\mathfrak{Bl}(a,b)$ is trivial in $\mathbb{Q}(t)/\mathbb{Q}[t,t^{-1}]$. Thus, for some $x \in \mathbb{Q}[t,t^{-1}]$, $\mathfrak{Bl}(a,b) = \left[\frac{x(t)}{\Delta(t)}\right]$ and $\mathfrak{Bl}^p(a_p,b_p) = \left[\frac{1}{\overline{r}s}\frac{x}{\Delta}\right]$.

If $\mathfrak{Bl}^p(a_p,b_p)$ is trivial in $\mathbb{Q}(t)/R_p$ then there is some $\dfrac{y_1}{y_2} \in R_p$ such that $\dfrac{1}{\overline{r}s}\dfrac{x}{\Delta} = \dfrac{y_1}{y_2}$ in $\mathbb{Q}(t)$. We cancel terms to see that $xy_2 = \overline{r}sy_1\Delta$. In particular, Δ divides xy_2. Since p (and so Δ) is relatively prime to y_2, it follows that Δ divides x and $\mathfrak{Bl}(a,b) = \left[\frac{x}{\Delta}\right] = 0$. Thus, $\mathfrak{Bl}(a_p,b_p) = 0$ if and only if $\mathfrak{Bl}(a,b) = 0$ and the characterization of part 2 of isotropic submodules follows.

Finally, consider the submodule $Q^p = QS_p^{-1}$. The kernel of the map $A_0(K) \to \frac{A_0^p(K)}{Q^p}$ is precisely Q, so that the map $f_{p,Q}$ factors as

$$\pi_1(M(K))^{(1)} \to A_0(K) \to \frac{A_0(K)}{Q} \hookrightarrow \frac{A_0^p(K)}{Q^p}.$$

Since $\frac{A_0(K)}{Q} \hookrightarrow \frac{A_0^p(K)}{Q^p}$ is a monomorphism, it does not affect the kernel of this composition. Thus, $\pi_1(M(K))^{(2)}_Q = \pi_1(M(K))^{(2)}_{p,Q^p}$ and the claim of part 2 about ρ-invariants follows. □

Using this property we show that the obstruction of Theorem 3.2.8 splits over connected sum of knots with coprime Alexander polynomials.

Theorem 3.3.2. *Let $K_1 \dots K_n$ be knots which have pairwise coprime Alexander polynomials and vanishing ρ^0-invariants. If $K = K_1\# \dots \#K_n$ is (1.5)-solvable, then for each i there an isotropic submodule $Q_i \subseteq A_0(K_i)$ with $\rho^1_{Q_i}(K_i) = 0$.*

Proof. By Theorem 3.2.8, for a symmetric polynomial, p, and $i = 1, \dots, n$, there is an isotropic submodule $Q_i^p \subseteq A_0^p(K_i)$ with

$$\sum_{i=1}^{n} \rho^1_{p,Q_i^p}(K_i) = 0 \tag{3.6}$$

Let p be the Alexander polynomial of K_i. By part 1 of Proposition 3.3.1 for $j \neq i$, all of the first order signatures of K_j localized at p are equal to $\rho^0(K_j)$ which vanishes by assumption. Dropping these terms from equation (3.6) leaves us with the conclusion that $\rho^1_{p,Q_i^p}(K_i) = 0$. By part 2 of Proposition 3.3.1, $Q_i^p = Q_i \otimes R_p$ for some isotropic submodule $Q_i \subseteq A_0(K_i)$ and $\rho^1_{Q_i}(K_i) = 0$. □

3.4 Anisotropic knots

In the case that a knot $K = K_1 \# K_2 \# \ldots \# K_n$ decomposes as a connected sum of knots for which $A_0^p(K_i)$ has no isotropic submodules, it turns out that the localized first order signatures do not depend on the choice of isotropic submodule. We will call a knot J <u>p-anisotropic</u> if $A_0^p(J)$ has no nontrivial isotropic submodules. If a knot is 0-anisotropic we simply call it isotropic.

Proposition 3.4.1. *Let p be a symmetric polynomial. If $K = K_1 \# \ldots \# K_n$ decomposes as a connected sum of p-anisotropic knots, then for every isotropic submodule $Q \subseteq A_0^p(K)$, $\rho^1_{p,Q}(K) = \rho^1_p(K)$.*

Proof. According to Proposition 3.2.7, there are isotropic submodules $Q_i \subseteq A_0^p(K_i)$ with $\rho^1_{p,Q}(K) = \sum_{i=1}^n \rho^1_{p,Q_i}(K_i)$. By assumption, there is only one isotropic submodule of $A_0^p(K_i)$, namely $Q_i = \{0\}$. Thus,

$$\rho^1_{p,Q}(K) = \sum_{i=1}^{n} \rho^1_p(K_i). \tag{3.7}$$

Notice that the right hand side of equation (3.7) is independent of Q. Thus, for any isotropic submodules $Q, Q' \subseteq A_0^p(K)$, $\rho^1_{p,Q}(K) = \rho^1_{p,Q'}(K)$. Taking $Q' = \{0\}$ completes the proof. $\square$

Making this substitution into Theorems 3.2.8 and Theorem 3.3.2, we see the following consequence.

Theorem 3.4.2. *If $K_1, \ldots, K_n$ are knots each of which decompose as a connected sum of p-anisotropic knots and $K = \#_{i=1}^n K_i$ is (1.5)-solvable, then $\rho^1_p(K) = \sum_{i=1}^n \rho^1_p(K_i) = 0$.*

If the knots $K_1, \ldots, K_n$ all have coprime Alexander polynomials and decompose as a connected sum of anisotropic knots and K is (1.5)-solvable, then $\rho^1(K_i) = 0$ for all i.

This theorem gives a powerful tool for the obstruction of linear dependences amongst knots which are anisotropic. We devote the remainder of this chapter to generating examples of p-anisotropic knots.

Proposition 3.4.3. *Let Δ be the Alexander polynomial of a knot, K. Suppose that every nonsymmetric prime factor of Δ and every prime factor of Δ with multiplicity greater than 1 is coprime to p . Then K is p-anisotropic.*

Proof. Since $\mathbb{Q}[t, t^{-1}]$ is a principal ideal domain (a PID) it follows that $A_0(K)$ has an elementary factor decomposition:

$$A_0(K) = \frac{\mathbb{Q}[t, t^{-1}]}{(q_1)} \oplus \frac{\mathbb{Q}[t, t^{-1}]}{(q_2)} \oplus \cdots \oplus \frac{\mathbb{Q}[t, t^{-1}]}{(q_n)}$$

where q_i divides q_{i+1} for all $1 \leq i \leq n-1$ and $q_1 q_2 \ldots q_n = \Delta$.

If q_i and p have a common prime factor, h, with $i < n$, then h also divides q_{i+1} and is a factor of Δ of multiplicity at least 2. This contradicts the assumption that high multiplicity prime factors of Δ are coprime to p. Thus, for $i < n$, $(q_i, p) = 1$ and so that $\frac{\mathbb{Q}[t,t^{-1}]}{(q_i)} \otimes R_p = 0$. Thus, $A_0^p(K) = A_0(K) \otimes R_p = \frac{R_p}{(q_n)}$. Since any factor of q_n which is not a factor of p is a unit in R_p, it follows that $A_0^p(K) \cong \frac{R_p}{(h)}$ with $h = (q_n, p)$ a factor of p. In particular, $A_0^p(K)$ is cyclic. Let x be a generator.

If there is some $y \in A_0^p(K)$ with $\mathfrak{Bl}^p(y,x) = 0$, then consider any $z \in A_0^p$. There is some $r \in R_p$ with $z = rx$. Then $\mathfrak{Bl}^p(y,z) = r\mathfrak{Bl}^p(y,x) = 0$. Since $\mathfrak{Bl}^p$ is nonsingular, this would imply that $y = 0$.

Now suppose that $Q \subseteq A_0^p(K)$ isotropic and $z = rx$ is some element of Q. Then, $0 = \mathfrak{Bl}^p(z,z) = \mathfrak{Bl}^p(x, \overline{r}rx)$. The previous paragraph then implies that $\overline{r}rx = 0$ and $\overline{r}r \in R_p$ must be a multiple of h. Since h is a common factor of p and Δ, h has no non-symmetric factors. Let g be a prime factor of h. Since it is symmetric, g dividing $r\overline{r}$ implies that g divides r. Since h has no factors of multiplicity greater than 1, every prime factor of h dividing r implies that h itself divides r. Thus $z = rx = 0$ in $A_0^p(K)$, Q contains only the zero element and K is p-anisotropic. $\square$

We will pull examples from the following corollary.

Corollary 3.4.4. *If the Alexander polynomial of K is squarefree and has no non-symmetric factors then K is p-anisotropic for every p.*

Chapter 4

Abelian ρ-invariants of links approximate first order signatures

In the following two chapters we provide the means we use to compute first order signatures. In this chapter we show that if a knot is algebraically slice then many of its first order signatures can be approximated by abelian signatures of a link. In Chapter 5 we show how to compute abelian signatures.

Let K be an algebraically slice knot bounding a genus g Seifert surface, Σ. By virtue of K being algebraically slice, there is a g-component link, L, sitting on Σ on which the linking form vanishes. The link L is called in [CHL10a] a derivative of K. At the end of this chapter we prove the intuitive fact that the submodule $Q(L) \subseteq A_0(K)$ generated by the lifts of the components of L is isotropic with respect to the Blanchfield form. Moreover, we show that $Q(L,p) := Q(L) \otimes R_p \subseteq A_0^p(K)$ is isotropic with respect to the localized Blanchfield form.

We need the following additional piece of notation:

Definition 4.0.5. For a knot K with Seifert surface Σ and a link L sitting on Σ, let γ be a component of L. A curve m in $S^3 - \Sigma$ is called a meridian for the band on which γ sits if m bounds a disk in S^3 which intersects Σ in an arc, intersects γ in a single point and does not intersect any other component of L.

We build a cobordism between $M(K)$ and $M(L)$ which we will use to relate their ρ-invariants.

Construction 4.0.6. Let Y_0 be given by starting with $M(K) \times [0,1]$ and and attaching a 2-handle to the zero framing of each component of L in $M(K) \times \{1\}$. After sliding K over these 2-handles as in Figure 4.1 it becomes apparent that K bounds a disk in $\partial_+ Y_0$. Thus, $\partial_+ Y_0$ is homeomorphic to $M(L) \# S^2 \times S^1$. Let Y be given by attaching a 3-handle to this non-separating 2-sphere. Thus, $\partial_- Y = M(K)$ while $\partial_+ Y = M(L)$.

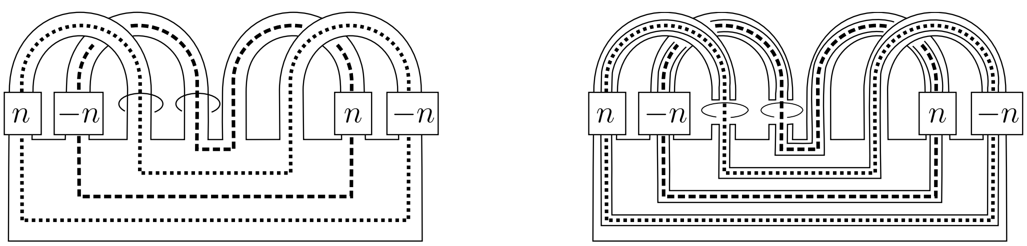

Figure 4.1: Right: a knot, a derivative, and meridians for the bands on which the derivative sits. Left: The knot becomes unknotted after sliding over the zero framing of each component of its derivative twice. The meridians to the bands are now meridians to the derivative.

We wish to use Y to make a claim involving $\rho^1_{p,Q(L,p)}(K)$ and a ρ-invariant of $M(L)$. We begin with an overview of the strategy. Let $\phi : \pi_1(Y) \to \frac{\pi_1(Y)}{\pi_1(Y)^{(2)}_p}$ be the projection map. (Notice that to even talk about this quotient, we must show that $H_1(Y) = \mathbb{Z}$.) Lemmas 4.1.1 and 4.1.2 will give that for some homomorphism from $\pi_1(M(L))$ to an abelian group, A,

$$\rho^1_{p,Q(L,p)}(K) - \rho(M(L), A) = \sigma^{(2)}(Y;\phi) - \sigma(Y). \tag{4.1}$$

Lemma 4.1.3 will give bounds on $\left|\sigma^{(2)}\left(Y; \frac{\pi_1(Y)}{\pi_1(Y)^{(2)}_p}\right) - \sigma(Y)\right|$.

4.1 Making the approximation.

Lemma 4.1.1. *1. The map induced by inclusion $H_1(M(K)) \to H_1(Y)$ is an isomorphism.*

2. The kernel of the map induced by inclusion from $A_0^p(K)$ to $A_0^p(Y)$ is equal to $Q(L,p)$.

Proof. In order to see the first claim, notice that Y is obtained from $M(K)$ by adding two handles along null-homologous curves and then adding a 3-handle. Neither of these operations change first homology.

Consider the Mayer Vietoris sequence with coefficients in R_p corresponding to the decomposition of Y_0 into $M(K) \times [0,1]$ together with g 2-handles.

$$H_1(N; R_p) \to H_1(M(K); R_p) \oplus H_1(\text{2-handles}; R_p) \to H_1(Y_0; R_p) \to 0,$$

where N is a regular neighborhood of L. Being a disjoint union of contractible spaces, $H_1(\text{2-handles}; R_p) = 0$. Since the components L are nullhomologous, the coefficient system to R_p is trivial on N and $H_1(N; R_p) \cong H_1(N; \mathbb{Z}) \otimes R_p \cong (R_p)^g$ is freely generated by lifts of the components of L. Thus, this exact sequence becomes

$$(R_p)^g \to H_1(M(K); R_p) \to H_1(Y_0; R_p) \to 0,$$

and $\ker(H_1(M(K); R_p) \to H_1(Y_0; R_p))$ is generated by the lifts of the components of L. Adding the remaining 3-handle does not change first homology, so that the kernel of the inclusion induced map $H_1(M(K); R_p) \to H_1(Y; R_p)$ is the submodule generated by the components of L. □

Lemma 4.1.2. *1. The map $H_1(M(L)) \to H_1(Y)$ is the zero map so that $M(L)$ lifts to the abelian cover of Y.*

2. *The map* $H_1(M(L)) \to A_0^p(Y) \cong \frac{A_0^p(K)}{Q(L,p)}$ *is given by sending* μ_k*, the meridian of the k'th component of* L *to the equivalence class of a lift of* m_k*.*

3. *The map* $H_1(M(L); R_p) \to A_0^p(Y)$ *is surjective.*

Proof. The meridian of the k'th component of L, μ_k, is isotopic in Y to m_k. These are nullhomologous in $M(K)$ and so in Y. This proves claim 1.

Again since μ_k isotopes to m_k, it follows that the map $H_1(M(L)) \otimes R_p \to A_0^p(Y)$ factors as

$$\begin{array}{ccccc} H_1(M(L)) & \to & A_0^p(K) & \to & \frac{A_0^p(K)}{Q(L,p)} \cong A_0^p(Y) \\ \mu_k & \mapsto & m_k, & & \end{array}$$

proving the second claim.

In order to see the second claim notice that by turing the relative handle-body structure of Y upside-down, Y can be obtained by instead staring with $M(L)$, adding a 1-handle and then g 2-handles. Let Y^1 be given by adding only the one-handle. Consider the long exact sequence of the pair $(Y, M(L))$ with coefficients in R_p

$$H_1(M(L); R_p) \to H_1(Y; R_p) \xrightarrow{p_*} H_1(Y, M(L); R_p) \xrightarrow{\partial_*} H_0(M(L); R_p) \xrightarrow{i_*} H_0(Y; R_p). \tag{4.2}$$

Since $H_1(M(L); \mathbb{Z}) \to H_1(Y; \mathbb{Z})$ is the zero map, $H_1(M(L); R_p) \cong H_1(M(L); \mathbb{Z}) \otimes R_p$ and $H_0(M(L); R_p) \cong R_p$. By [Coc04, Proposition 3.7] $H_0(Y; R_p)$ is torsion. In particular, $H_0(M(L); R_p) \to H_0(Y; R_p)$ cannot be injective. Since R_p is a PID, it follows that $\ker(i_*) \cong R_p$.

The pair $(Y, M(L))$ has only one relative 1-handle. Thus, $H_1(Y, M(L); R_p)$ is cyclic. Using the classification of modules over a PID, it follows that $H_1(Y, M(L); R_p)$ is either free of rank 1 or is torsion. Since $H_1(Y, M(L); R_p)$ surjects onto the rank 1 free-module $\text{im}(\partial_*) = \ker(i_*) \cong R_p$, it cannot be torsion. Again, since R_p is a PID, Any epimorphism from R_p to itself is also a monomorphism. Thus, ∂_* is a monomorphism, p_* is the zero map, and $H_1(M(L);R_p) \to H_1(Y;R_p)$ is an epimorphism, proving the final claim. $\square$

Theorems 4.1.1 and 4.1.2 imply

$$\sigma^{(2)}\left(Y, \frac{\pi_1(Y)}{\pi_1(Y)^{(2)}_p}\right) - \sigma(Y) = \rho^1_{p,Q(L,p)}(K) - \rho(M(L),A), \tag{4.3}$$

where A is the subgroup of $\frac{A_0^p(K)}{Q(L,p)}$ generated by $m_1,\ldots m_g$. Being a finitely generated subgroup of a torsion free abelian group, A is free abelian.

Lemma 4.1.3 bounds the signatures on the left hand side of this equation. Before we can state this lemma, we must provide a definition for the Alexander nullity of a link. Let $\phi : \pi_1(M(L)) \twoheadrightarrow A$ be an epimorphism onto the free-abelian group A. We will refer to the $\mathbb{Q}[A]$-rank of $H_1(M(L);\mathbb{Q}[A])$ as the nullity of L with respect to A and denote it by $\eta(L,A)$.

Lemma 4.1.3. *Let Y be the 4-manifold given by Construction 4.0.6. Consider the quotient map $\pi_1(Y) \to \frac{\pi_1(Y)}{\pi_1(Y)^{(2)}_p}$.*

1. $\sigma(Y) = 0$.
2. $\left|\sigma^{(2)}\left(Y, \frac{\pi_1(Y)}{\pi_1(Y)^{(2)}_p}\right)\right| \leq g - 1 - \eta(L,A)$ *where A is the abelian group given by the submodule of $\frac{A_0^p(K)}{Q(L,p)}$ generated by $m_1,\ldots m_g$.*

Proof. To see the first claim notice that the homology of Y which is not carried by $M(K)$ is generated by 2-handles attached to $M(K)$ along the zero framings of g curves which have zero linking numbers with each other. The intersection form is thus given by the $g \times g$ zero matrix, which proves (1).

In the proof of the second claim, we begin with the assumption that the composition

$$\pi_1(M(L)) \to \pi_1(Y) \to \frac{\pi_1(Y)}{\pi_1(Y)^{(2)}_p} \tag{4.4}$$

is nontrivial.

Let $\mathcal{K}$ be the classical skew field of fractions of the Ore domain $\mathbb{Q}\left[\frac{\pi_1(Y)}{\pi_1(Y)_p^{(2)}}\right]$. To see the second claim we show that

$$\mathrm{rank}_{\mathcal{K}}\left(\frac{H_2\left(Y;\mathcal{K}\right)}{i_*\left[H_2\left(\partial Y;\mathcal{K}\right)\right]}\right) = g-1-\eta(L,A).$$

The pair $(Y, M(K))$ consists of g relative 2-handles and 1 relative 3-handle so $\chi(Y) - \chi(M(K)) = g-1$. Since $M(K)$ is a closed 3-manifold, $\chi(M(K)) = 0$. It must be that $\chi(Y) = g-1$. By [Coc04, Proposition 3.7], $H_0(Y;\mathcal{K}) = 0$. By [Coc04, Proposition 3.10], $H_1(Y;\mathcal{K}) = 0$. Y has the homotopy type of a 3-complex, so $H_4(Y;\mathcal{K}) = 0$. Consider the long exact sequence of the pair $(Y, \partial Y)$,

$$H_3(\partial Y;\mathcal{K}) \to H_3(Y;\mathcal{K}) \to H_3(Y,\partial Y;\mathcal{K}).$$

Employing Poincaré duality and the universal coefficient theorem over the skew field $\mathcal{K}$, $H_3(\partial Y;\mathcal{K}) = H_0(\partial Y;\mathcal{K}) = 0$ and $H_3(Y,\partial Y;\mathcal{K}) = H_1(Y;\mathcal{K}) = 0$. Thus, $H_3(Y;\mathcal{K}) = 0$. Thus, $H_n(W;\mathcal{K}) = 0$ for all $n \neq 2$. The alternating sum of the ranks of twisted homology gives the Euler characteristic, so $\mathrm{rank}_{\mathcal{K}}\left(H_2(Y;\mathcal{K})\right) = \chi(Y) = g-1$.

Since $A \subseteq \frac{\pi_1(Y)}{\pi_1(Y)_p^{(2)}}$, $\mathcal{K}$ is free and so flat over the field $\mathbb{Q}(A)$,

$$H_1(\partial_+ Y;\mathcal{K}) \cong H_1(M(L);\mathbb{Q}(A)) \otimes \mathcal{K} \cong \mathcal{K}^{\eta(L,A)}.$$

Since $H_1(\partial_- Y;\mathcal{K}) = 0$, it follows that $H_1(\partial Y;\mathcal{K}) \cong \mathcal{K}^{\eta(L,A)}$. By Poincaré duality, $H_2(\partial Y;\mathcal{K}) \cong H_1(\partial Y;\mathcal{K}) \cong \mathcal{K}^{\eta(L,A)}$. Since $H_3(Y,\partial Y;\mathcal{K}) = 0$, the exact sequence of the pair indicates that $i_* : H_2(\partial Y; \mathcal{K}) \to H_2(Y; \mathcal{K})$ is a monomorphism. Thus,

$$\begin{aligned} \operatorname{rank}_{\mathcal{K}}\left(\frac{H_2(Y;\mathcal{K})}{i_*\left[H_2(\partial Y;\mathcal{K})\right]}\right) &= \operatorname{rank}_{\mathcal{K}}\left(H_2(Y;\mathcal{K})\right) - \operatorname{rank}_{\mathcal{K}}\left(H_2(\partial Y;\mathcal{K})\right) \\ &= (g-1) - \eta(L,A). \end{aligned}$$

Finally, $\left|\sigma^{(2)}(Y)\right| \leq \operatorname{rank}_{\mathcal{K}}\left(\frac{H_2(Y;\mathcal{K})}{H_2(\partial Y;\mathcal{K})}\right)$ by inequality (2.1), completing the proof in the case that the composition (4.4) is nontrivial.

If it happens that (4.4) is trivial, then it follows that the map $H_1(M(L); R_p) \to H_1(Y; R_p)$ is the zero homomorphism. By part 3 of 4.1.2, this would imply that $H_1(Y; R_p) = 0$, $\pi_1(Y)^{(2)}_p = \pi_1(Y)^{(1)}$ and $\frac{\pi_1(Y)}{\pi_1(Y)^{(2)}_p} = \frac{\pi_1(Y)}{\pi_1(Y)^{(1)}}$. Definition 2.2.1 now implies that

$$\sigma^{(2)}\left(Y, \frac{\pi_1(Y)}{\pi_1(Y)^{(2)}_p}\right) = \rho(M(K);\mathbb{Z}) = \rho^0(K).$$

Recall that K is algebraically slice so that $\rho^0(K) = 0$. □

Combining the results of this section, namely equation (4.3) and Lemma 4.1.3 we get the following approximation theorem:

Theorem 4.1.4. *Suppose that a knot K is algebraically slice and has L as a g-component derivative. Let $Q(L,p)$ be the submodule of $A^p_0(K)$ generated by the lifts of the components of L. Let A be the subgroup of $A^p_0(K)/Q(L,p)$ generated by the meridians of the bands on which L sits. Then*

$$|\rho^1_{p,Q}(L) - \rho(M(L),A)| \leq g - 1 - \eta(L,A)$$

The submodule $Q(L,p) \subseteq A^p_0(K)$ generated by the derivative L is isotropic, as is proven in the next section. in the case that K has only p-anisotropic prime factors Proposition 3.4.1 implies that $\rho^1_{p,Q(L,p)}(K) = \rho^1_p(K)$ and gives the following corollary.

Corollary 4.1.5. *If a knot K is algebraically slice and has only p-anisotropic prime factors, and has L as a g-component derivative, let $Q(L,p)$ be the submodule of $A_0^p(K)$ generated by the lifts of the components of L. Let A be the subgroup of $A_0^p(K)/Q(L,p)$ generated by the meridians of the bands on which L sits. Then*

$$|\rho_p^1(L) - \rho(M(L), A)| \leq g - 1 - \eta(L, A).$$

4.2 Submodules generated by derivatives are isotropic

In this section we show that the submodule of $A_0^p(K)$ generated by the lifts of a derivative of K is isotropic. The proof relies on the formula in [Kea75, section 8] for the Blanchfield form in terms of the Seifert matrix.

Lemma 4.2.1. *Suppose that Σ is a genus g Seifert surface for a knot K and $L = L_1 \dots L_g$ is a g-component link sitting on Σ which spans a rank g direct summand of $H_1(\Sigma)$ on which the Seifert form vanishes. The submodule $Q(L,p) \subseteq A_0^p(K)$ generated by the components of L is isotropic.*

Proof. The proof will proceed by showing that the submodule of $A_0(K)$ generated by L is isotropic. Once this is done then Proposition 2.4.5 will complete the proof in the localized setting. Indeed, for any $\alpha \otimes \frac{e}{f}, \beta \otimes \frac{g}{h}$ in the submodule of $Q(L,p)$,

$$\mathfrak{Bl}^p\left(\alpha \otimes \frac{e}{f}, \beta \otimes \frac{g}{h}\right) = \Psi(\mathfrak{Bl}(\alpha,\beta))\frac{\overline{e}g}{\overline{f}h}. \tag{4.5}$$

If we can only show that the unlocalized module $Q(L) \subseteq A_0(K)$ generated by L is isotropic then we will conclude that $Q(L,p)$ is isotropic.

Let the set $\{L_1 \dots L_g\}$ be extended to $\{L_1 \dots L_g, D_1, \dots D_g\}$, a symplectic basis for $H_1(\Sigma)$. Let $\mu_1, \dots \mu_g, \nu_1, \dots, \nu_g$ be the dual basis for $H_1(S^3 - \Sigma)$ given by meridians about the bands on which L_i and D_i sit. The homology classes of the lifts of

$\mu_1, \dots \mu_g, \nu_1, \dots, \nu_g$ to the infinite cyclic cover of $M(K)$ form a generating set for $A_0(K)$ as a $\mathbb{Q}$-vector space. The map from $H_1(\Sigma)$ to the Alexander module induced by lifting Σ to the cyclic cover of $M(K)$ is given with respect to these generating sets by the Seifert matrix V. The Blanchfield form with respect to the generating set given by the lifts of $\mu_1, \dots \mu_g, \nu_1, \dots, \nu_g$ is given by

$$\mathfrak{Bl}(\vec{r}, \vec{s}) = (1-t)(\vec{s})^T (V - tV^T)^{-1} (\vec{r}) \tag{4.6}$$

(see [Kea75, section 8]).

Since $\{L_1 \dots L_g\}$ is a metabolizer for the Seifert form, V is given by a matrix of the form $\begin{bmatrix} 0 & A \\ B & C \end{bmatrix}$, with respect to the basis $\{L_1 \dots L_g, D_1, \dots D_g\}$ for $H_1(\Sigma)$ (A, B, C are $g \times g$ matrices). Thus, $(V - tV^T)^{-1}$ is a matrix with entries in $\mathbb{Q}[t^{\pm 1}]$ of the form $\begin{bmatrix} D(t) & E(t) \\ F(t) & 0 \end{bmatrix}$, where $D(t), E(t), F(t)$ are $g \times g$ matrices with rational function entries.

Consider any $\vec{r} = V \begin{bmatrix} \vec{a} \\ \vec{0} \end{bmatrix}$, $\vec{s} = V \begin{bmatrix} \vec{b} \\ \vec{0} \end{bmatrix}$ in $Q(L)$ (a and b are g-dimensional column vectors and $\vec{0}$ denotes the g-dimensional zero vector). Plugging these values into (4.6) we see

$$\mathfrak{Bl}(\vec{r}, \vec{s}) = (1-t) \begin{bmatrix} \vec{b} \\ \vec{0} \end{bmatrix}^T \begin{bmatrix} 0 & B^T \\ A^T & C^T \end{bmatrix} \begin{bmatrix} D & E \\ F & 0 \end{bmatrix} \begin{bmatrix} 0 & A \\ B & C \end{bmatrix} \begin{bmatrix} \vec{a} \\ \vec{0} \end{bmatrix}.$$

This is zero by direct computation. Thus, $Q(L)$ is isotropic as claimed. $\square$

Chapter 5

Abelian ρ-invariants of links via the Cimasoni-Florens signature

The previous chapter, namely Theorem 4.1.4, allows us to approximate first order signatures of a knot in terms of abelian ρ-invariants associated to derivatives. In order to take advantage of this approximation, we need a tool which will allow us to compute ρ-invariants of zero surgery along links with zero linking numbers corresponding to maps to abelian groups. In order to build such a tool, we make use of a signature function defined by Cimasoni and Florens in [CF08].

5.1 Background: The Cimasoni-Florens signature function and unitary signatures

In [CF08], Cimasoni and Florens define a signature function for colored links. In this section we recall their construction.

An <u>n-colored link</u> $L = L(1), \ldots, L(n)$ is a link whose every component is decorated with an integer between 1 and n. The notation $L(k)$ refers to the sublink of L consisting of components colored with the integer k.

Let $\mathbb{T}^n$ denote the n-dimensional unit torus in $\mathbb{C}^n$, $\mathbb{T}^n := \{(\omega_1, \ldots, \omega_n) \in \mathbb{C}^n : |\omega_k| = 1 \text{ for all } k\}$. The Cimasoni-Florens signature of an n-colored link L, σ_L is an integer valued function on $\mathbb{T}^n_* := \{(\omega_1, \ldots, \omega_n) \in \mathbb{T}^n : \omega_k \neq 1 \text{ for all } k\}$.

5.1.1 C-complexes, linking forms and the signature function

We begin by giving the definition of a C-Complex for an n-colored link in [CF08].

Definition 5.1.1 (Subsecton 2.1 of [CF08]). A union of embedded possibly non-disjoint surfaces $S = S_1 \ldots S_n$ is a C-complex (or Clasp-Complex) for the n-colored link, $L = L(1) \ldots L(n)$, if the following conditions hold:

1. For each i S_i is a Seifert surface for $L(i)$. That is, S_i is a compact oriented embedded surface with no closed components whose oriented boundary is $L(i)$.

2. For $i \neq j$, $S_i \cap S_j$ is a union of arcs each having one boundary point in $\partial S_i = L(i)$ and the other in $\partial S_j = L(j)$. These intersections are called clasps. See the leftmost part of Figure 5.1.

3. For distinct numbers i, j, k, $S_i \cap S_j \cap S_k$ is empty.

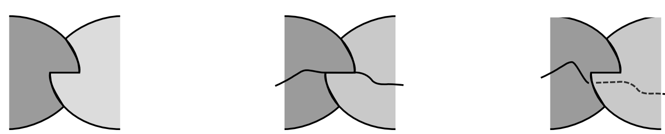

Figure 5.1: Left to Right: (1) A clasp in a C-Complex. (2) A loop γ passing through a clasp. (3) The curve γ^{+-} gotten by pushing γ in the positive normal direction on one component on the C-complex and in the negative normal direction of the other.

It is worth noting that every colored link has a C-complex. In [CF08], Cimasoni and Florens define a matrix with Laurent polynomial entries coefficients in terms of a C-complex. We present an abbreviated version of their construction.

A simple closed curve in F is called a loop if, whenever it intersects a clasp, it does so as in the second picture in Figure 5.1. Let $B = \{\gamma_1, \dots, \gamma_m\}$ be a basis for $H_1(F)$ given by loops.

For a loop $\gamma_j \in B$ and $\epsilon = \langle \epsilon_1 \dots \epsilon_n \rangle \in \{\pm 1\}^n$ define γ_j^ϵ by pushing the portion of γ_j carried by S_i off of S_i in the positive normal direction if $\epsilon_i = +1$ and in the negative normal direction otherwise (see the third picture in Figure 5.1). Let A^ϵ be the matrix whose (i,j)-entry is $\operatorname{lnk}(\gamma_i^\epsilon, \gamma_j)$. Define $H(\omega_1, \dots, \omega_n)$ by

$$H(\omega_1, \dots, \omega_n) = \sum_{\epsilon \in \{\pm 1\}^n} (1 - \omega_1^{-\epsilon_1}) \dots (1 - \omega_n^{-\epsilon_n}) A^\epsilon. \tag{5.1}$$

The matrix H is Hermitian for each $\omega \in \mathbb{T}^n$. They define the following isotopy invariant.

Definition 5.1.2. [Subsecton 2.2 of [CF08]] The Cimasoni-Florens signature function of the n-colored link L, $\sigma_L : \mathbb{T}_*^n \to \mathbb{Z}$ is defined at $\omega \in \mathbb{T}_*^n$ to be the signature of $H(\omega)$, that is,

$$\sigma_L(\omega) := \sigma(H(\omega)) = \#\{\text{positive eigenvalues}\} - \#\{\text{negative eigenvalues}\}.$$

In [CF08], Cimasoni and Florens prove that σ_L is independent of the C-Complex F and the basis B, that is, σ_L is a color preserving isotopy invariant of L.

5.1.2 Unitary signatures and ρ-invariants

In this Chapter we make use of another signature invariant of 4-manifolds, the twisted unitary signature.

For a complex vector space $V \cong \mathbb{C}^n$, $U(V) \cong U(n)$ denotes the set unitary linear transformations V. From a representation $\alpha : \Gamma \to U(V)$, V inherits the structure of a $\mathbb{Z}[\Gamma]$-module. We denote this module structure by $V_\alpha \cong \mathbb{C}^n_\alpha$.

For a CW-complex, W, and a representation $\alpha : \pi_1(W) \to U(n)$, the twisted homology $H_*(W; \mathbb{C}^n_\alpha)$ is defined similarly to the twisted homology of Section 2.3. Let $\widetilde{X}$ be the universal cover of X and consider the tensored chain complex

$$(C_*(X; \mathbb{C}^n_\alpha), \partial) = \Big[\ldots \overset{\partial_{k+1} \otimes 1}{\to} C_k(\widetilde{X}) \otimes \mathbb{C}^n_\alpha \overset{\partial_k \otimes 1}{\to} C_{k-1}(\widetilde{X}) \otimes \mathbb{C}^n_\alpha \overset{\partial_{k-1} \otimes 1}{\to} \ldots \Big].$$

Also similarly to as in Section 2.3 if $X = W$ is a compact oriented 4-manifold, there is a Hermitian form $Q^\alpha : H_2(W; \mathbb{C}^n_\alpha) \times H_2(W; \mathbb{C}^n_\alpha) \to \mathbb{C}$ called the intersection form. $H_2(W; \mathbb{C}^n_\alpha)$ decomposes as a direct sum of spaces on which Q^α is positive definite $H_2^+(W; \mathbb{C}^n_\alpha)$, negative definite, $H_2^-(W; \mathbb{C}^n_\alpha)$, and zero, $H_2^0(W; \mathbb{C}^n_\alpha)$. The twisted signature of W is defined as $\sigma(W, \alpha) := \dim(H_2^+(W; \mathbb{C}^n_\alpha)) - \dim(H_2^-(W; \mathbb{C}^n_\alpha))$.

The difference $\sigma(W, \alpha) - n\sigma(W)$ depends only on $\alpha|_{\partial W}$ and one can use this signature invariant to define a ρ-invariant of 3-manifolds. More precisely, let M be a 3-manifold and $\alpha : \pi_1(M) \to U(n)$ be a representation such that there exists a compact oriented 4-manifold W with $\partial W = M$ and a representation $\alpha : \pi_1(W) \to U(n)$ making the following diagram commute.

$$\begin{tikzcd}
\pi_1(M) \arrow[r, "\alpha"] \arrow[d] & U(n) \\
\pi_1(W) \arrow[ur, "\alpha"] &
\end{tikzcd}$$

Then $\rho(M, \alpha) = \sigma(W, \alpha) - n\sigma(W)$. Not every unitary representation of the fundamental group of a 3-manifold extends over a 4-manifold in this manner. This definition is not technically complete. Every unitary representation in which we are interested in this chapter will extend. This definition is adequate for our purposes.

5.2 One dimensional unitary and abelian ρ-invariants and the Cimasoni-Florens signature

Before we begin we will need one more piece of notation. Let $A = \langle g_1 \dots g_n | r_1 \dots r_m \rangle$ with $r_k = \sum_{j=1}^{n} a_{k,j} g_j ,=$ and $a_{k,j} \in \mathbb{Z}$ be a presentation for the abelian group, A. Let $\mathbb{T}_A$ be defined by

$$\mathbb{T}_A := \left\{ (z_1, \dots, z_n) \in \mathbb{T}^n : \text{for all } k, \prod_{j=1}^{n} z_j^{a_{k,j}} = 1 \right\}$$

Notice that the map sending g_i to the 1×1 matrix $[z_i] \in U(1)$ extends to a homomorphism $A \to U(1)$ if and only if $(z_1 \dots z_n) \in \mathbb{T}_A$.

In this chapter we prove the following theorem realizing ρ-invariants of zero surgery on a link with zero linking numbers as an integral of a function $\widehat{\sigma}_L : \mathbb{T}^n \to \mathbb{Z}$ which agrees with the Cimasoni-Florens signature of L on $\mathbb{T}^n_*$. The definition of the extension $\widehat{\sigma}_L$ is given in Definition 5.3.6. We will be considering n-component links, $L = L_1, \dots L_n$ with no given coloring. We will regard these as n-colored links by taking $L(k) = L_k$. The goal of this chapter is the following theorem, to be proven in Section 5.4

Theorem 5.4.3. *Let $A = \langle g_1 \dots g_n | r_1 \dots r_m \rangle$ be an abelian group. Let $L = L_1, \dots L_n$ be an n-component link with zero pairwise linking numbers. Let $\phi : H_1(M(L)) \to A$ be given by sending μ_i to g_i. Then*

$$\rho(M(L), \phi) = \frac{1}{\lambda(\mathbb{T}_A)} \int_{\mathbb{T}_A} \widehat{\sigma}_L(\omega) d\lambda(\omega),$$

where λ is Lebesgue measure on $\mathbb{T}_A$.

In Section 5.3 we prove the following theorem, which realizes the unitary ρ-invariants of $M(L)$ by $\widehat{\sigma}$.

Theorem 5.3.7. *For $\omega = (\omega_1, \ldots, \omega_n) \in \mathbb{T}^n_{\mathbb{Q}}$, the representation α_ω from $\pi_1(M(L))$ to the subgroup of $U(1)$ given by mapping μ_i to $[\omega_i]$, $\rho(M(L), \alpha_\omega) = \widehat{\sigma}_L(\omega)$*

We are interested in L^2-ρ-invariants associated to maps to abelian groups and unitary ρ-invariants associated to maps to $U(1)$, which is also abelian. Thus, we begin by building a 4-manifold, W, bounded by $M(L)$ together with a disjoint union of product 3-manifolds such that any map from $\pi_1(M(L))$ to an abelian group, A, extends to a map $\pi_1(W) \to A$.

Before we give the construction we need the following lemma about links with zero pairwise linking.

Lemma 5.2.1. *Let $L = L_1, \ldots L_n$ be an n-component link with zero pairwise linking. There exist disjoint compact oriented properly embedded surfaces $F_1, \ldots, F_n \subseteq B^4$ with $\partial F_i = L_i$.*

Proof. Since $\operatorname{lnk}(L_i, L_j) = 0$ for all $j \neq i$, there is a Seifert surface $F_i \subseteq S^3$ for L_i which is disjoint from all L_j. Thus we find a (possibly non-disjoint) collection of surfaces $F_1, \ldots, F_n$ with $F_i \cap L_j = \emptyset$ for all $i \neq j$. By pushing the interior of F_i a different distance into B^4 than we push the interior of F_j, we get the desired conclusion $\square$

Construction 5.2.2. For link $L = L_1 \ldots L_n$ with zero pairwise linking, let $F = F_1 \ldots F_n$ be a collection of n embedded disjoint surfaces in B^4 such that $\partial F_i = L_i$. Let $E(F)$ denote the exterior of F in B^4. The exterior of L, $E(L)$, is a submanifold of $\partial E(F)$. Along a neighborhood of each component of the boundary of $E(L)$ (a torus in S^3 enclosing a component of L) attach to $E(F)$ a copy of $S^1 \times B^2 \times I$ so that the nullhomologous longitudes of the components of L bound disks. Call the resulting 4-manifold W.

The boundary of W consists of a copy of $M(L)$ together with the orientation reverse of $\widehat{F} \times S^1$ where $\widehat{F} = \widehat{F_1} \sqcup \cdots \sqcup \widehat{F_n}$ denotes the n-component closed surface given by capping the only boundary curve of each component of F with a disk.

The following lemma will be used to compute the ρ-invariants of $M(L)$:

Lemma 5.2.3. *Let W be the 4-manifold given in Construction 5.2.2. Then*

1. *the inclusion induced map $H_1(M(L)) \to H_1(W)$ is an isomorphism*

2. *$\sigma(W) = 0$*

Notice that as a consequence Lemma 5.2.3 we have the following commutative diagram for any one dimensional unitary representation $\alpha : \pi_1(M(L)) \to U(1)$. Since $U(1)$ is abelian we may as well regard α as a representation of $H_1(M(L))$.

$$\begin{tikzcd}
\pi_1(M(L)) \arrow[r] \arrow[d] & H_1(M(L)) \arrow[r, "\alpha"] \arrow[d, "\cong"'] & U(1) \\
\pi_1(W) \arrow[r] & H_1(W) \arrow[ur, "\alpha"] &
\end{tikzcd}$$

Thus,

$$\rho(M(L), \alpha) - \sum_{i=1}^{n} \rho(\widehat{F_i} \times S^1, \alpha) = \sigma(W, \alpha) - \sigma(W) = \sigma(W, \alpha). \tag{5.2}$$

Once we have proven this lemma, it will remain only to analyze the twisted signatures of W and the one dimensional unitary ρ-invariants of product 3-manifolds. This is done in Section 5.3. Similarly, any homomorphism to an abelian group $\phi : \pi_1(M(L)) \to A$ extends over W and

$$\rho(M(L), \phi) - \sum_{i=1}^{n} \rho(\widehat{F_i} \times S^1, \phi) = \sigma^{(2)}(W, \phi). \tag{5.3}$$

In Section 5.4 we compute $\sigma^{(2)}(W, \phi)$ and $\rho(\widehat{F_i} \times S^1, \phi)$.

Proof of Lemma 5.2.3. An easy argument using the long exact sequence of the pair

$(B^4, B^4 - F)$ and duality can be used to show that the inclusion induced map $H_1(E(L)) \to H_1(E(F))$ is an isomorphism. Since the longitudes are nullhomologous, the inclusion induced maps $H_1(E(L)) \to H_1(M(L))$ and $H_1(E(F)) \to H_1(W)$ are isomorphisms. It follows that the bottom row of the following commutative diagram is an isomorphism.

$$\begin{tikzcd} H_1(E(L)) \arrow[r, "\cong"] \arrow[d, "\cong"'] & H_1(E(F)) \arrow[d, "\cong"'] \\ H_1(M(L)) \arrow[r] & H_1(W) \end{tikzcd}$$

In order to see (2), first observe that if A and B are classes in $H_2(E(F))$ whose intersection pairing (A, B) is nonzero and i_* denotes the inclusion induced map $H_2(E(F)) \to H_2(B^4)$, then $(i_*(A), i_*(B)) = (A, B) \neq 0$. However, $H_2(B^4) = 0$, so this is impossible. The intersection matrix for $E(F)$ must be the zero matrix. Since the longitudes of L are nullhomologous, $H_2(W) = H_2(E(F)) \oplus \mathbb{Z}^n$. The $\mathbb{Z}^n$-factor has a basis given by the embedded surfaces $\widehat{F}_i$. These embedded surfaces sit in the boundary of W and so have zero intersection numbers. Thus, the intersection matrix for W is the zero matrix, $H_2^+(W) = H_2^-(W) = 0$, and $\sigma(W) = 0$. $\square$

5.3 One dimensional unitary ρ-invariants

In this section we prove Theorem 5.3.7. In light of equation (5.2) it remains only to analyze the one dimensional unitary signatures of the 4-manifold W of Construction 5.2.2 and the one dimensional unitary ρ-invariants of the product 3-manifold $F \times S^1$

First we recall a result of Cimasoni-Florens.

Theorem ([CF08], Theorem 6.1). *Let $\omega = (\omega_1, \ldots, \omega_n) \in \mathbb{T}^n_* \cap \mathbb{T}^n_{\mathbb{Q}}$ be an n-tuple of roots of unity none of which are equal to* 1*. Let $L = L_1, \ldots, L_n$ be an n-component link* *with zero pairwise linking. Regard L as an n-colored link by coloring each component differently. For a union of n disjoint, compact, connected, oriented, properly embedded surfaces $F = F_1 \cup \cdots \cup F_n \subseteq B^4$ with $\partial F_i = L_i$ and the representation $\alpha_\omega : \pi_1(E(F)) \to U(1)$ sending μ_i, the meridian of L_i, to $[\omega_i]$, $\sigma(E(F), \alpha_\omega) = \sigma_L(\omega)$.*

Remark 5.3.1. As stated and proven in [CF08], the above theorem requires neither zero linking numbers nor that all components have distinct colors. The statement we give is as strong as needed in our setting

We prove the following propositions at the end of this section.

Proposition 5.3.2. *For the 4-manifold W of Construction 5.2.2 and $\omega \in \mathbb{T}^n_*$, the inclusion induced map $H_2(E(F); \mathbb{C}_{\alpha_\omega}) \to H_2(W; \mathbb{C}_{\alpha_\omega})$ is an isomorphism and hence $\sigma(W, \alpha_\omega) = \sigma(E(F), \alpha_\omega)$.*

Proposition 5.3.3. *For a closed oriented connected surface F, and a representation $\alpha : \pi_1(F \times S^1) \to U(1)$ with image in the roots of unity, $\rho(F \times S^1, \alpha) = 0$*

Putting this all together:

$$\begin{aligned}
\rho(M(L), \alpha_\omega) &= \sigma(W, \alpha_\omega) - \textstyle\sum_i \rho(\widehat{F}_i \times S^1, \alpha_\omega) && \text{by Equation (5.2)} \\
&= \sigma(W, \alpha_\omega), && \text{by Proposition 5.3.3} \\
&= \sigma(E(F), \alpha_\omega) \text{ if } \omega \in \mathbb{T}^n_* && \text{by Proposition 5.3.2} \\
&= \sigma_L(\omega) \text{ if } \omega \in \mathbb{T}^n_* \cap \mathbb{T}^n_{\mathbb{Q}} && \text{by [CF08, Theorem 6.1].}
\end{aligned}$$

This proves Theorem 5.3.7 in the case that no ω_i is equal to 1. The main difficulty in completing the proof is that the Cimasoni-Florens signature is not defined when $\omega_i = 1$. As the following proposition indicates, even in this setting the Cimasoni-Florens signature (of a different colored link) computes these unitary ρ-invariants.

Proposition 5.3.4. *Let $L = L_1 \ldots L_n$ be an n-component link with zero pairwise linking. Let $\omega \in \mathbb{T}^n_{\mathbb{Q}}$ be an n-tuple of roots of unity. Let $A = \{k : \omega_k = 1\}$ and*

$m = |A|$ be the cardinality of A. Let J be the $(n+m)$-component link given by replacing L_k by two parallel copies of L_k with opposite orientations.

Regard J as an n-colored link by taking $J(k) = L_k$ if $k \notin A$ and $J(k)$ to be the two parallel copies of L_k if $k \in A$. Let $\omega^J \in \mathbb{T}^n_{\mathbb{Q}} \cap \mathbb{T}^n_*$ be an n-tuple of roots of unity gotten from ω by replacing ω_k with any root of unity other than 1 for each $k \in A$.

Then $\rho(M(L), \alpha_\omega) = \sigma_J(\omega^J)$

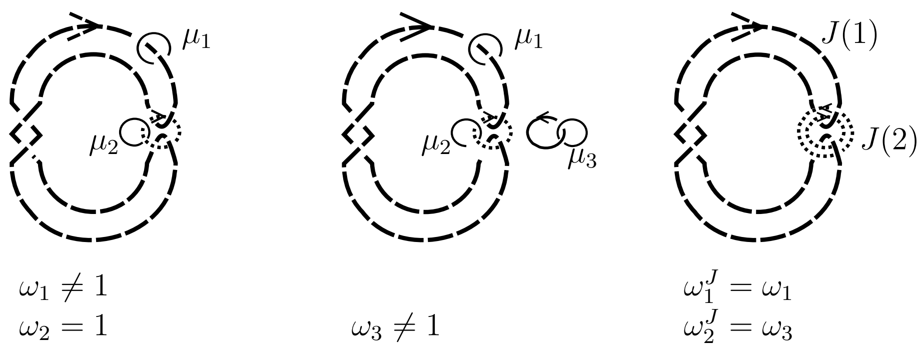

Figure 5.2: Left to right: (1) The two component link L. The map α_ω is trivial on the the meridian of the second component of L. (2) The link L_0 gotten by adding an unknotted component for each ω_k equal to 1. (3) The colored link J is gotten from L_0 by a zero framed handleslide. Proposition 5.3.4 claims that $\rho(M(L), \alpha_\omega) = \sigma_J(\omega^J)$

Proof. Let $L^0 = L_1 \cup \cdots \cup L_n \cup U_1 \cup \cdots \cup U_m$ be the $(n+m)$-component link given by the split union of L with an m-component unlink. Regard it as an $(n+m)$-component link by taking $L^0(k) = L_k$ if $k \leq n$ and $L^0(n+k) = U_k$.

Let ω^0 be an $(n+m)$-tuple of roots of unity given by $(\omega_1, \ldots, \omega_n, z_1, \ldots, z_m)$ where $(z_1, \ldots, z_m) \in \mathbb{T}^m_{\mathbb{Q}} \cap \mathbb{T}^m_*$ is an m-tuple of roots of unity other than 1. Von Neumann ρ-invariants add under connected sum of 3-manifolds, $M(L^0) = M(L)\#(\#_m S^2 \times S^1)$, and the one dimensional unitary ρ-invariants of $S^2 \times S^1$ vanish (for example, by Proposition 5.3.3). It follows that

$$\begin{aligned} \rho(M(L^0), \alpha_{\omega^0}) &= \rho(M(L), \alpha_\omega) + \sum_{i=1}^{m} \rho(S^1 \times S^2, \alpha_{z_i}) \\ &= \rho(M(L), \alpha_\omega). \end{aligned} \tag{5.4}$$

Let $A = \{a_1, \ldots, a_m\}$. Consider the $(n+m)$-component, $(n+m)$-colored link, J^0, is gotten from L^0 by sliding U_k over $-L_{a_k}$ for $k = 1, \ldots, m$. Let J be as in the statement of Proposition 5.3.4. Notice that the uncolored links gotten from J^0 and J by forgetting about their colorings are isotopic. Kirby calculus (see [GS99]) provides a homeomorphism ϕ between the zero surgeries $M(L^0)$ and $M(J^0) = M(J)$. The map induced by ϕ on the level of first homology is given by

$$\begin{array}{c} H_1(M(L^0)) = \langle \mu_1, \ldots \mu_n, \nu_1, \ldots, \nu_m \rangle \xrightarrow{\phi_*} \\ H_1(M(J^0)) = \langle \{\mu_j\}_{j \notin A}, \{\mu^1_{a_k}, \mu^2_{a_k}\}_{a_k \in A} \rangle \end{array}$$

$$\begin{aligned} \mu_j &\mapsto \mu_j \text{ if } j \notin A \\ \mu_j &\mapsto \mu^1_j - \mu^2_j \text{ if } j \in A \\ \nu_j &\mapsto \mu^2_{a_j}, \end{aligned}$$

where μ_j is the meridian of L_j, ν_j is the meridian of U_j, and $\mu^1_{a_k}$ and $\mu^2_{a_k}$ are the meridians of the two parallel copies of L_{a_k}. Thus, the following diagram commutes:

$$\begin{tikzcd} H_1(M(L^0)) \arrow[r, "\alpha^0_\omega"] \arrow[d, "\phi_*"'] & U(1) \\ H_1(M(J)) \arrow[ur, "\alpha'"'] & \end{tikzcd}$$

where α' is defined by

$$\begin{aligned} \alpha'(\mu_j) &= [\omega_j] \quad \text{if } j \notin A \\ \alpha'(\mu^1_{a_k}) &= [z_k] \\ \alpha'(\mu^2_{a_k}) &= [z_k]. \end{aligned}$$

By design, $\alpha' : \pi_1(M(J)) \to U(1)$ is realized as $\alpha_{\omega'}$ with $\omega' \in \mathbb{T}^{n+m}_* \cap \mathbb{T}^{n+m}_{\mathbb{Q}}$. Thus, the case of Theorem 5.3.7 already proven applies and

$$\rho(M(L), \alpha_\omega) = \rho(M(J^0), \alpha_{\omega'}) = \sigma_{J^0}(\omega').$$

Recall the following property of the Cimasoni-Florens signature.

Proposition 5.3.5 (Proposition 2.5, [CF08]). *Let $L = L(1) \cup \cdots \cup L(n+1)$ be an $(n+1)$-colored link and $L' = L'(1) \cup \cdots \cup L'(n)$ be the n-colored link with $L'(k) = L(k)$ for $k < n$ and $L'(n) = L(n) \cup L(n+1)$. For any $\omega = (\omega_1, \ldots, \omega_n) \in \mathbb{T}^n_*$,*

$$\sigma_{L'}(\omega_1, \ldots, \omega_{n-1}, \omega_n) = \sigma_L(\omega_1, \ldots, \omega_{n-1}, \omega_n, \omega_n) - \operatorname{lnk}(L(n), L(n+1))$$

where $\operatorname{lnk}(L(n), L(n+1))$ *is the total linking number between the sublinks $L(n)$ and $L(n+1)$.*

The n-colored link J is gotten from J^0 by identifying the $(a_k)^{\text{th}}$ and $(n+k)^{\text{th}}$ colors. The $(a_k)^{\text{th}}$ and $(n+k)^{\text{th}}$ entries of ω' are both equal to z_k. The n-tuple ω^J is obtained by deleting the final m terms from ω'. Since J^0 has zero pairwise linking numbers, Proposition 5.3.5 implies that $\sigma_{J_0}(\omega') = \sigma_J(\omega^J)$ and completes the proof. $\square$

In light of Proposition 5.3.4, we make the following extension of the definition of $\sigma_L(\omega)$.

Definition 5.3.6. Let L be a n-component, n-colored link with zero pairwise linking. Let ω be an n-tuple of roots of unity. Let $A = \{a : \omega_a = 1\}$. Let $m = |A|$. Let J be the $(m+n)$-component, n-colored link with $J(k) = L(k)$ if $k \notin A$ and $J(k)$ given by two parallel pushoffs of $L(k)$ with opposite orientations if $k \in A$. Let $\omega' \in \mathbb{T}^n_*$ be given by $\omega'_k = \omega_k$ if $k \notin A$ and $\omega'_k \neq 1$ to be any nontrivial root of unity when $k \in A$. Define $\widehat{\sigma}_L(\omega) := \sigma_{L'}(\omega')$.

Rephrasing Proposition 5.3.4 in terms of $\widehat{\sigma}_L$, we have the following theorem.

Theorem 5.3.7. *For L an n-component link and $\omega = (\omega_1, \ldots, \omega_n) \in \mathbb{T}^n_{\mathbb{Q}}$. Let α_ω be the representation from $\pi_1(M(L))$ to the subgroup of $U(1)$ given by mapping μ_i to $[\omega_i]$. Then $\rho(M(L), \alpha_\omega) = \widehat{\sigma}_L(\omega)$.*

We end this section with the proofs of Propositions 5.3.2 and 5.3.3.

Proof of Theorem 5.3.2. Consider the Mayer-Vietoris long exact sequence twisted by α_ω corresponding to the decomposition of W as the union of $E(F)$ together with n disjoint copies of $S^1 \times B^2 \times I$ glued together along n disjoint copies of $S^1 \times S^1 \times I$:

$$\begin{gathered} \bigoplus_{i=1}^{n} H_2(S^1 \times S^1 \times I; \mathbb{C}_{\alpha_\omega}) \to \\ H_2(E(F); \mathbb{C}_{\alpha_\omega}) \oplus \left(\bigoplus_{i=1}^{n} H_2(S^1 \times D^2 \times I; \mathbb{C}_{\alpha_\omega}) \right) \\ \to H_2(W; \mathbb{C}_{\alpha_\omega}) \to \bigoplus_{i=1}^{n} H_1(S^1 \times S^1 \times I; \mathbb{C}_{\alpha_\omega}) \end{gathered} \tag{5.5}$$

The twisted chain complex for the i^{th} copy of $S^1 \times D^2 \times I \sim S^1$ is chain homotopy equivalent to

$$\mathbb{C} \xrightarrow[{[1-\omega_i]}]{} \mathbb{C} \ ,$$

while the twisted chain complex for the i^{th} copy of $S^1 \times S^1 \times I \sim S^1 \times S^1$ is chain homotopy equivalent to

$$\mathbb{C} \xrightarrow[{\begin{bmatrix} 0 \\ 1-\omega_i \end{bmatrix}}]{} \mathbb{C}^2 \xrightarrow[{\begin{bmatrix} 1-\omega_i & 0 \end{bmatrix}}]{} \mathbb{C} \ .$$

Both of these chain complexes are acyclic when $\omega_i \neq 1$. Thus, $H_p(S^1 \times S^1 \times I; \mathbb{C}_\omega)$ and $H_p(S^1 \times B^2 \times I; \mathbb{C}_\omega)$ both vanish for all p and the exact sequence of (5.5) reveals that $H_2(E(F); \mathbb{C}_\omega) \to H_2(W; \mathbb{C}_\omega)$ is an isomorphism. $\square$

We now prove Proposition 5.3.3, showing that the unitary ρ-invariants of product 3-manifolds vanish. This will complete the proof of Theorem 5.3.7.

Proof of Proposition 5.3.3. Let $M = F \times S^1$ and s denote the element of $H_1(M)$ given by the S^1-factor.

First we deal with the case that $\alpha(s) = [1]$ is trivial. In this case α factors as

$H_1(F \times S^1) \to H_1(F \times B^2) \xrightarrow{\overline{\alpha}} U(1)$. Since $F \times B^2$ deformation retracts to a subset of its boundary it follows that the inclusion induced maps $H_2(F \times S^1) \to H_2(F \times B^2)$ and $H_2(F \times S^1, \mathbb{C}_\alpha) \to H_2(F \times B^2, \mathbb{C}_\alpha)$ are epimorphisms and the twisted and untwisted signatures vanish. Thus, $\rho(F \times S^1, \alpha) = \sigma(F \times B^2, \overline{\alpha}) - \sigma(F \times B^2) = 0$.

We now assume that $\alpha(s) \neq [1]$ is nontrivial. If F has genus at least two, let γ be an essential separating curve. Let W_0 be the 3-manifold given by adding to $F \times [0,1]$ a two handle along $\gamma \times \{1\}$. Then $\partial W_0 = F \sqcup F_1 \sqcup F_2$, where F_1 and F_2 have genus strictly less than the genus of F. Since γ is nullhomologous in F, the inclusion induced maps $H_1(F) \xrightarrow{\cong} H_1(W_0)$ and $H_1(F_1) \oplus H_1(F_2) \xrightarrow{\cong} H_1(W_0)$ are isomorphisms. Let $W = W_0 \times S^1$. The Künneth formula guarantees that the map $H_1(F \times S^1) \to H_1(W)$ is an isomorphism. Thus, α extends over $\pi_1(W)$ and restricts to representations of $\pi_1(F_1 \times S^1)$ and $\pi_1(F_2 \times S^1)$. Thus, $\rho(M, \alpha) - \rho(F_1 \times S^1, \alpha) - \rho(F_2 \times S^1, \alpha) = \sigma(W, \alpha) - \sigma(W)$.

Alternately, W_0 can be obtained from $F_1 \sqcup F_2$ by adding a one-handle between different components. For $k = 0, 1, 2$, the inclusion induced maps $H_k(F_1 \sqcup F_2) \twoheadrightarrow H_k(W_0)$ is onto. By the Künneth formula then $H_2(F_1 \times S^1 \sqcup F_2 \times S^2) \twoheadrightarrow H_2(W)$ is surjective so that $\sigma(W) = 0$.

We now compute $\sigma(W, \alpha)$. Consider the Mayer Vietoris long exact sequence corresponding to the decomposition $W = (F \times S^1) \cup (\text{2-handle} \times S^1)$ with $(F \times S^1) \cap (\text{2-handle} \times S^1) = N(\gamma) \times S^1$ for $N(\gamma)$ a neighborhood of γ.

$$\begin{aligned} &H_2(N(\gamma) \times S^1; \mathbb{C}_\alpha) \to H_2(F \times S^1; \mathbb{C}_\alpha) \oplus H_2(\text{2-handle} \times S^1; \mathbb{C}_\alpha) \\ &\qquad \to H_2(W; \mathbb{C}_\alpha) \to H_1(N(\gamma) \times S^1; \mathbb{C}_\alpha) \end{aligned} \tag{5.6}$$

Since 2-handle $\times S^1 \sim S^1$ is a homotopy one complex, it follows that $H_2(D^2 \times S^1 \times I; \mathbb{C}_\alpha) = 0$. The chain complex for $N(\gamma) \times S^1 \sim S^1 \times S^1$ twisted by α is chain homotopic to

$$\mathbb{C} \xrightarrow{\begin{pmatrix} 0 \\ 1-\alpha(s) \end{pmatrix}} \mathbb{C}^2 \xrightarrow{\begin{pmatrix} 1-\alpha(s) & 0 \end{pmatrix}} \mathbb{C}.$$

Since $\alpha(s) \neq 1$, this sequence is acyclic and $H_*(N(\gamma)\times S^1; \mathbb{C}_\alpha) = 0$. The sequence (5.6) reduces to the conclusion that $H_2(F \times S^1; \mathbb{C}_\alpha) \to H_2(W; \mathbb{C}_\alpha)$ is an isomorphism so that $\sigma(W, \alpha) = 0$.

Thus, $\rho(F \times S^1, \alpha) - \rho(F_1 \times S^1, \alpha) - \rho(F_2 \times S^1, \alpha) = 0$ and an argument inducting on the genus of F will complete the proof once we have proven the Lemma in the case that F is a torus, $\mathbb{T}^2$, or a sphere, S^2.

If $F = S^2$, then α extends over $S^1 \times B^3$, a homotopy 1-complex. Thus, $\rho(S^2 \times S^1, \alpha) = \sigma(S^1 \times B^3, \alpha) - \sigma(S^1 \times B^3) = 0$.

If $F = \mathbb{T}^2$ then $M = \mathbb{T}^3$ is a 3-torus. Since every finitely generated subgroup of the roots of unity is finite cyclic, α factors as

$$\pi_1(M) \cong \mathbb{Z}^3 \overset{\alpha_0}{\twoheadrightarrow} \mathbb{Z}_p \hookrightarrow U(1).$$

For some basis, $\{b_1, b_2, b_3\}$, for $\pi_1(M)$, $\alpha_0(b_1) = \alpha_0(b_2) = 1$ are trivial and $\alpha_0(b_3)$ is a generator of $\mathbb{Z}_p$. Let f_* be the automorphism of $\pi_1(M)$ sending the basis $\{s_1, s_2, s_3\}$ coming from the product structure of $M = S^1 \times S^1 \times S^1$ to $\{b_1, b_2, b_3\}$. Since M is an Eilenberg-MacClane space, there is a homotopy equivalence $f_0 : M \to M$ inducing f_*. By [Hem04, Corollary 13.7], f_0 is homotopic to a homeomorphism, f. Thus, we have the following commutative diagram,

$$\begin{tikzcd} \pi_1(M) \arrow[r, "\widehat{\alpha}"] \arrow[d, "f_*"'] & U(1) \\ \pi_1(M) \arrow[ur, "\alpha"'] & \end{tikzcd}$$

where $\widehat{\alpha} = \alpha \circ f_*$. Since the ρ-invariant is a homeomorphism invariant of 3-manifolds, $\rho(M, \alpha) = \rho(M, \widehat{\alpha})$. Since $\widehat{\alpha}(s_1) = [1]$ is trivial, $\rho(M, \widehat{\alpha}) = 0$ by the case with which we opened the proof. $\square$

We close with an observation which will be necessary in the analysis in Section 5.4.

Corollary 5.3.8. *For a link L with zero linking number and $\omega \in \mathbb{T}^n_{\mathbb{Q}}$, $\sigma(W, \alpha_\omega) = \widehat{\sigma}_L(\omega)$*

Proof. By equation (5.2), $\sigma(W, \alpha_\omega)$ is given by the difference between $\rho(M(L), \alpha_\omega)$ and a sum of one dimensional unitary ρ-invariants of product manifolds. By Proposition 5.3.3, $\sigma(W, \alpha_\omega) = \rho(M(L), \alpha_\omega)$. Theorem 5.3.7 now implies the result. $\square$

5.4 Computing the abelian L^2-signatures of W

We begin by recalling a result of Lück-Schick [LS05, Theorem 0.1]. They prove that if Y is a 4-manifold and $\phi : \pi_1(Y) \to \Lambda$ is a homomorphism to a residually finite group, then $\sigma^{(2)}(Y, \Lambda)$ can be approximated by signatures corresponding to homomorphisms to finite groups. If $\Lambda = \Lambda_0 \geq \Lambda_1 \geq \dots$, where Λ_i is normal and finite index in Λ and $\Lambda_0 \cap \Lambda_1 \cap \dots = 0$, then taking $p_k : \Lambda \to \Lambda/\Lambda_k$ to be the quotient map,

$$\sigma^{(2)}(Y, \phi) = \lim_{k \to \infty} \sigma^{(2)}(Y, p_k \circ \phi). \tag{5.7}$$

In the case that Λ is finite, $l^2(\Lambda) = \mathbb{C}[\Lambda]$ and the von Neumann dimension $\dim_{\mathcal{N}(\Lambda)}$ used to define the L^2-signature gives the same information as the classical dimension theory (see [Lüc09, example 1.14]). To be precise, for a finitely generated $\mathbb{C}[\Lambda]$-module V, $\dim_{\mathcal{N}(\Lambda)}(V) = \frac{1}{|\Lambda|}\dim(V)$, where $\dim(V)$ is the dimension of V as a $\mathbb{C}$-vector space. Unwinding this fact gives $\sigma^{(2)}(Y,\phi) = \frac{1}{|\Lambda|}\sigma(\widetilde{Y}_\phi)$.

Additionally, in the case that Λ is finite, a classical fact of representation theory says that $\mathbb{C}[\Lambda]$ is the direct sum of irreducible unitary representations of Λ [Ser77, Section 6.2 Proposition 10]. That is, if $X = \{\alpha : \Lambda \to U(V_\alpha)\}$ is the set of all irreducible unitary representations of Λ, then $\mathbb{C}[\Lambda]$ is isomorphic as a $\mathbb{C}[\Lambda]$-module to $\bigoplus_{\alpha\in X} V_\alpha$. In particular, each V_α is projective and hence flat, and

$$H_2(\widetilde{Y}_\phi;\mathbb{C}) = H_2(Y;\mathbb{C}[\Lambda]) = H_2\left(Y;\bigoplus_{\alpha\in X} V_\alpha\right) = \bigoplus_{\alpha\in X} H_2(Y;V_\alpha).$$

Moreover, the intersection form on $H_2(\widetilde{Y}_\phi)$ splits as the direct sum of the intersection forms on $H_2(Y;V_\alpha)$, so that

$$\sigma^{(2)}(Y,\phi) = \frac{1}{|\Lambda|}\sigma(\widetilde{Y_\phi}) = \frac{1}{|\Lambda|}\sum_\alpha \dim(V_\alpha)\sigma(Y,\alpha\circ\phi). \tag{5.8}$$

Between equations (5.7) and (5.8), we see that the unitary signatures of a 4-manifold determine its finite and residually finite L^2-signatures.

Using this mindset together with Corollary 5.3.8, we express the L^2-signatures of W (from Construction 5.2.2) as a limit of Riemann sums for an integral of $\widehat{\sigma}_L$.

Proposition 5.4.1. *Let $A = \langle g_1,\dots,g_n | r_1,\dots,r_m\rangle$ be a finitely generated Abelian group. Let W be the 4-manifold of Construction 5.2.2. Let $\phi : H_1(W) \to A$ be given by sending μ_i to g_i, then*

$$\sigma^{(2)}(W,\phi) = \frac{1}{\lambda(\mathbb{T}_A)}\int_{\mathbb{T}_A}\widehat{\sigma}_L(\omega)d\lambda(\omega) \tag{5.9}$$

where λ is Lebesgue measure on $\mathbb{T}_A$.

Proof. We begin with the case that A is a finite abelian group. In this case equation (5.8) together with the fact that $\mathbb{T}_A$ parametrizes all unitary representations of A gives that

$$\sigma^{(2)}(W,\phi) = \frac{1}{|\mathbb{T}_A|}\sum_{\omega\in\mathbb{T}_A}\sigma(W,\alpha_\omega).$$

By Corollary 5.3.8, $\sigma(W,\alpha_\omega) = \widehat{\sigma}_L(\omega)$, which gives

$$\sigma^{(2)}(W,\phi) = \frac{1}{|\mathbb{T}_A|}\sum_{\omega\in\mathbb{T}_A}\widehat{\sigma}_L(\omega). \tag{5.10}$$

When A is finite, λ is counting measure and integration against λ is summation. After these substitutions, equation (5.10) becomes equation (5.9), completing the proof in the case the A is finite abelian.

Next, for any finitely generated abelian group A, A has a resolution by finite index subgroups $A \geq A_1 \geq A_2 \geq \dots$ where $A_k = k!A = \langle k!g_1,\dots,k!g_n\rangle$. (Recall that $k! = k(k-1)\dots(3)(2)(1)$ denotes "k-factorial".) The quotient, A/A_k, has a presentation

$$A/A_k = \langle g_1,\dots,g_n,|r_1,\dots,r_m,k!g_1,\dots,k!g_n\rangle$$

and $\mathbb{T}_{A/A_k} = \mathbb{T}_A \cap \{(\omega_1,\dots,\omega_n) : \omega_1^{k!} = \dots = \omega_n^{k!} = 1\} \subseteq \mathbb{T}^n_{\mathbb{Q}}$.

Thus, by (5.7) and (5.8), for a homomorphism $\phi : \pi_1(W) \to A$ and the projection map $p_k : A \to A/A_k$,

$$\begin{aligned}\sigma^{(2)}(W,\phi) &= \lim_{k\to\infty}\sigma(W,p_k\circ\phi)\\ &= \lim_{k\to\infty}\frac{1}{|\mathbb{T}_{A/A_k}|}\sum_{\omega\in\mathbb{T}_{A/A_k}}\widehat{\sigma}_L(\omega).\end{aligned}$$

This realizes $\sigma^{(2)}(W,A)$ as a limit of Riemann sums for $\frac{1}{\lambda(\mathbb{T}_A)}\int_{\mathbb{T}_A}\widehat{\sigma}_L(\omega)d\lambda(\omega)$. By [CF08, Theorem 4.1] $\widehat{\sigma}_L$ is piecewise continuous, and so Riemann sums converge to the integral. This completes the proof. □

The following proposition shows that a great many ρ-invariants of product 3-manifolds vanish. It is much stronger than is needed for our applications. We hope that this result is of independent interest.

Proposition 5.4.2. *For a closed oriented surface F and a homomorphism to a residually finite group (for example a finitely generated abelian group) $\phi : \pi_1(F \times S^1) \to H$, $\rho(F \times S^1, \phi) = 0$.*

Before proving Proposition 5.4.2 we observe that as a consequence we can compute the abelian L^2-ρ-invariants of zero-surgery on links via the Cimasoni-Florens signature function.

Theorem 5.4.3. *Let $A = \langle g_1 \dots g_n | r_1 \dots r_m \rangle$ be an abelian group. Let L be an n-component link with zero pairwise linking numbers. Let $\phi : H_1(M(L)) \to A$ be given by sending μ_i to g_i, then*

$$\rho(M(L), \phi) = \frac{1}{\lambda(\mathbb{T}_A)} \int_{\mathbb{T}_A} \widehat{\sigma}_L(\omega) d\lambda(\omega)$$

where λ is Lebesgue measure on $\mathbb{T}_A$.

Proof. By equation (5.3), $\rho(M(L), \phi) = \sigma^{(2)}(W, \phi) + \sum \rho(\widehat{F}_i \times S^1, \phi)$. Then by Propositions 5.4.1 and 5.4.2,

$$\rho(M(L), \phi) = \frac{1}{\lambda(\mathbb{T}_A)} \int_{\mathbb{T}_A} \widehat{\sigma}_L(\omega) d\lambda(\omega).$$

□

We close the chapter with the proof of Proposition 5.4.2.

Proof of Proposition 5.4.2. The von Neumann ρ-invariant has the property that for a 3-manifold X, groups A and B, a homomorphism $\phi : \pi_1(X) \to A$ and a monomorphism $\psi : A \hookrightarrow B$, $\rho(X, \phi) = \rho(X, \psi \circ \phi)$. (See [COT03, Proposition 5.13].) Thus, by replacing H by $\mathrm{im}(\phi)$, we may assume that ϕ is onto.

Let s denote the generator of $\pi_1(S^1)$ so that $\pi_1(F \times S^1) = \pi_1(F) \oplus \langle s \rangle$. Let $\phi(s) = h$. Since s is central in $\pi_1(F \times S^1)$ and ϕ is onto, h is central in H, that is, h commutes with every element of H. The proof proceeds in three steps:

1. We express $\rho(F \times S^1, \phi)$ as a limit of ρ-invariants corresponding to homomorphisms to finite groups.
2. We reduce the computation of $\rho(F \times S^1, \phi)$ for a homomorphism ϕ from $\pi_1(F \times S^1)$ to a finite group G to the case that $G \cong \mathbb{Z}_n$ is finite cyclic.
3. We show that $\rho(F \times S^1, \phi) = 0$ when the target of ϕ is a finite cyclic group.

Step 1: Let Σ be a once puntured torus and $\{a, b\}$ be a symplectic basis for Σ. Let $E = \pi_1(\Sigma)$ be the free group on a and b. Let X be the 4-manifold $F \times \Sigma$. Consider the group,

$$G = \frac{H \oplus E}{h = [a, b]}$$

where $[a, b] = aba^{-1}b^{-1}$. Denote the obvious map $\pi_1(X) = \pi_1(F) \oplus E \to G$ by $\overline{\phi}$. On the boundary, this factors through ϕ. Since h is central in H it does so injectively. That is, the following diagram commutes

$$\begin{tikzcd}
\pi_1(F \times S^1) \arrow[r, "\phi"] \arrow[d] & H \arrow[d, hook] \\
\pi_1(F \times \Sigma) \arrow[r, "\overline{\phi}"] & G
\end{tikzcd}$$

and $\rho(F \times S^1, \phi) = \sigma^{(2)}(F \times \Sigma, \overline{\phi}) - \sigma(F \times \Sigma)$. We hope to use the behavior of residually finite L^2-signatures to compute $\sigma^{(2)}(F \times \Sigma, \overline{\phi})$. We must show that G is residually finite.

Lemma 5.4.4. *Let $E = \langle a, b | \rangle$ be the rank two free group with generators a and b. Let H be a residually finite group. Let h be an element of the center of H. Then*

$$G = \frac{H \oplus E}{h = [a, b]}$$

is residually finite.

Proof. We begin by noticing that in G, both a and b commute with $h = [a, b]$, so that in G, both of $[a, [a, b]] =]$ and $[b, [a, b]]$ are trivial so that

$$G \cong \frac{H \oplus E'}{h = [a, b]}$$

where

$$E' = \langle a, b | [a, [a, b]] = [b, [a, b]] = 1 \rangle$$

is isomorphic to the 3-dimensional integral Heisenberg group. That is, the multiplicative group of 3×3 upper triangular integral matrices with ones on the main diagonal,

$$E' \cong \left\{ \begin{bmatrix} 1 & \alpha & \gamma \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{bmatrix} : \alpha, \beta, \gamma \in \mathbb{Z} \right\}. \tag{5.11}$$

For an integer p, let $\phi_p : E' \to E_p \leq Gl(n, \mathbb{Z}_p)$ be the homomorphism given by reduction of entries mod p. In this finite quotient, $\phi_p([a, b])$ is order p.

Let g be a nontrivial element of G. We will find a finite quotient of G in which g is still non-trivial. If we can do so we will conclude that G is residually finite.

Notice that g is represented by an equivalence class $(x \oplus y)$, $x \in H$, $y \in E'$. If y

does not sit in the cyclic subgroup generated by $[a,b]$ in E' (which is normal), then $(x \oplus y)$ is nonzero in the quotient,

$$q : G \to \frac{E'}{([a,b])} \cong \mathbb{Z}^2,$$

whose codomain is residually finite. In some finite quotient of $\mathbb{Z}^2$, $q(g) = q(x \oplus y)$ is nonzero.

Alternately, if $y = [a,b]^m$ for some $m \in \mathbb{Z}$, then $g = (x \oplus [a,b]^m) = (xh^m \oplus 1)$ in G. If $(x \oplus y)$ is nontrivial, then $x' = xh^m$ is nontrivial in H, which by assumption is residually finite. Let $f : H \to \overline{H}$ be a homomorphism to a finite group with $f(x') \neq 1$. Let n be the order of $f(h)$ in this group. The direct sum $f \oplus \phi_n : H \oplus E' \to \overline{H} \oplus E_n$ passes to a homomorphism

$$F : \frac{H \oplus E'}{h = [a,b]} \to \frac{\overline{H} \oplus E_n}{f(h) = \phi_n([a,b])}.$$

Consider the resulting commutative diagram

$$\begin{tikzcd}
H \arrow[r] \arrow[d, "f"] & H \oplus E' \arrow[r] \arrow[d, "f\oplus\phi_n"] & G \cong \dfrac{H \oplus E'}{h = [a,b]} \arrow[d, "F"] \\
\overline{H} \arrow[r] & \overline{H} \oplus E_n \arrow[r] & \dfrac{\overline{H} \oplus E_n}{f(h) = \phi_n([a,b])}
\end{tikzcd}$$

Since h is central in H, $f(h)$ is central in $\overline{H}$ and the composition along the bottom row is injective. Since by assumption $f(x') \neq 1$, it follows that $F(g) = F(x' \oplus 1) \neq 1$. Thus, every nontrivial element of G is nontrivial in some finite quotient and G is residually finite, as we claimed.

$\square$

Let $G \geq G_1 \geq G_2 \geq \ldots$ where G/G_k is finite and $G_1 \cap G_2 \cap \cdots = \emptyset$. Take $p_k : G \to G/G_k$ be the quotient map.

$$\begin{aligned}
\rho(F \times S^1, \phi) &= \sigma^{(2)}(F \times \Sigma, \overline{\phi}) - \sigma(F \times \Sigma) \\
&= \lim_{k \to \infty} \left(\sigma^{(2)}(F \times \Sigma, p_k \circ \overline{\phi}) - \sigma(F \times \Sigma)\right) \\
&= \lim_{k \to \infty} \rho(F \times S^1, p_k \circ \overline{\phi}).
\end{aligned}$$

Since $p_k \circ \overline{\phi} : \pi_1(F \times S^1) \to G/G_k$ is a homomorphism to a finite group for each k, it remains only to prove the theorem in the case that H finite.

Step 2: Assuming that H is a finite group, h^n is trivial in H for some n. Let P be the n times punctured sphere. The group, $\pi_1(P)$ is isomorphic to $\langle s_1 \ldots s_n | s_1 s_2 \ldots s_n = 1\rangle$. Consider the homomorphism

$$\begin{aligned}
&\overline{\phi} : \pi_1(F \times P) \cong \pi_1(F) \oplus \langle s_1 \ldots s_n | s_1 s_2 \ldots s_n = 1\rangle \to H \\
&\overline{\phi}(f) = \phi(f) \text{ if } f \in \pi_1(F) \\
&\overline{\phi}(s_i) = h \text{ for } i = 1, \ldots, n
\end{aligned}$$

Notice that $\partial(F \times P) = F \times \partial P$ is given by n copies of $F \times S^1$. On each of these boundary components, $\overline{\phi}$ restricts to ϕ. Thus, $n\rho(F \times S^1, \phi) = \sigma^{(2)}(F \times P, \overline{\phi}) - \sigma(F \times P)$. By an argument based on the Künneth theorem, the map $H_2(F \times \partial P) \to H_2(F \times P)$ is surjective and $\sigma(F \times P) = 0$.

Since H is finite, the L^2-signature and the signature of the $\overline{\phi}$-cover agree,

$$\sigma^{(2)}(F \times P, \overline{\phi}) = \frac{1}{|H|} \sigma(\widetilde{F \times P}_{\overline{\phi}}).$$

In order to compute $\sigma(\widetilde{F \times P}_{\overline{\phi}})$, we first study the cover of $F \times P$ corresponding to the composition

$$\psi : \pi_1(F \times P) \xrightarrow{\overline{\phi}} H \to \frac{H}{\langle h \rangle}.$$

Recall that h is central in H, so the cyclic subgroup it generates is normal in H and this quotient makes sense. Since ψ is trivial on $\pi_1(P)$, the corresponding cover is given by the product of a (compact) cover of F with P, $\widetilde{F}_\psi \times P$. A compact cover of a closed oriented surface is still a closed oriented surface.

Since ψ factors through ϕ, the covering map corresponding to ϕ factors through the covering map corresponding to ψ. Consider the resulting tower of covers.

$$\begin{tikzcd}
\widetilde{F \times P}_{\overline{\phi}} \arrow[dd] \arrow[dr, "p"] & \\
& \widetilde{F}_\psi \times P \arrow[dl] \\
F \times P &
\end{tikzcd}$$

The group of deck translations of p is isomorphic to

$$\frac{\pi_1(\widetilde{F}_\psi \times P)}{p_*[\pi_1(\widetilde{F \times P})_{\overline{\phi}}]} \cong \frac{\ker(\psi)}{\ker(\overline{\phi})} \cong \overline{\phi}[\ker(\psi)] = \ker\left(H \to \frac{H}{\langle h \rangle}\right) = \langle h \rangle \cong \mathbb{Z}_n$$

Thus, $\widetilde{F \times P}_{\overline{\phi}}$ is a finite cyclic cover of $\widetilde{F}_\psi \times P$. It remains then only to show that for a closed oriented surface $F' = \widetilde{F_\psi}$ and a homomorphism ϕ' from $\pi_1(F' \times P) = \pi_1(F') \oplus \langle s_1 \ldots s_n | s_1 s_2 \ldots s_n = 1 \rangle$ to a finite cyclic group $\mathbb{Z}_n$ which sends each s_i to the same generator of $\mathbb{Z}_n$, the signature of the resulting cover is zero.

Consider the following chain of equalities:

$$\begin{aligned}
\sigma\left(\widetilde{F' \times P}_{\phi'}\right) &= n\sigma^{(2)}(F' \times P, \phi') && \text{by the first equality of (5.8)} \\
&= n(\sigma^{(2)}(F' \times P, \phi') - \sigma(F' \times P)) && \text{since } \sigma(F' \times P) = 0 \\
&= n\rho(F' \times S^1, \phi') && \text{by the definition of } \rho\text{-invariants.}
\end{aligned}$$

It remains only to show that for a homomorphism $\phi' : \pi_1(F' \times S^1) = \pi_1(F') \oplus \langle s \rangle \to \mathbb{Z}_n$ sending s to a generator of $\mathbb{Z}_n$, $\rho(F' \times S^1, \phi') = 0$.

Step 3: Pick a basis for $\pi_1(F')$, $\{a_i, b_i\}_{i=1}^{g(F')}$. Since $\phi'(s)$ generates $\mathbb{Z}_m$, $\phi(a_i) = \phi'(s)^{p_i}$ and $\phi(b_i) = \phi'(s)^{q_i}$ for some $p_i, q_i \in \mathbb{Z}$. Consider the automorphism of $\pi_1(F' \times S^1)$ given by $s \mapsto s$, $a_i \mapsto a_i s^{-p_i}$ and $b_i \mapsto b_i s^{-q_i}$. Similarly to the analysis in the proof of Proposition 5.3.3, this map is induced by a homeomorphism $\Phi : F' \times S^1 \to F' \times S^1$.

Thus, $\rho(F' \times S^1, \phi') = \rho(F' \times S^1, \phi' \circ \Phi_*)$, and $\phi_0 := \phi' \circ \Phi_*$ is trivial on $\pi_1(F')$. Let V be a handlebody bounded by F'. Since ϕ_0 is trivial on $\pi_1(F')$, ϕ_0 extends to a homomorphism $\overline{\phi_0} : \pi_1(V \times S^1) \to \mathbb{Z}_n$. There is a deformation retraction from V to a subset of F' so that both of

$$H_2(F' \times S^1) \twoheadrightarrow H_2(V \times S^1) \text{ and } H_2(F' \times S^1; \mathbb{C}[\mathbb{Z}_n]) \twoheadrightarrow H_2(V \times S^1; \mathbb{C}[\mathbb{Z}_n])$$

are epimorphisms and the signature and L^2-signature vanish and $\rho(F' \times S^1, \phi_0) = \sigma^{(2)}(V \times S^1, \overline{\phi_0}) - \sigma(V \times S^1) = 0$. This completes the proof of Lemma 5.4.2 $\square$

Chapter 6

Application to the linear independence of the twist knots

In this section, we study those twist knots, T_n, (depicted in Figure 1.1) which are algebraically of order 2. We briefly outline the strategy. Theorem 3.4.2 provides an obstruction to these knots being linearly dependent. Corollary 4.1.5 shows that this obstruction is approximated by abelian ρ-invariants of derivatives. Theorem 5.4.3 gives us a means to compute these ρ-invariants.

We recall the algebraic concordance classification of the twist knots given by Levine in [Lev69].

1. For $n < 0$, T_n is of infinite order in the algebraic concordance group.

2. T_n is algebraically slice if an only if $n = a^2 + a$ for some $a \in \mathbb{Z}$.

3. All remaining twist knots are of order either 2 or 4 in the algebraic concordance group.

4. A linear combination $\underset{n=-m}{\overset{m}{\#}} a_n T_n$ is algebraically slice if and only if $a_n T_n$ is algebraically slice for all n.

Amongst the algebraically slice twist knots, the 0 and 2-twist knots are slice. Casson and Gordon in [CG78] show that these are the only slice twist knots. Jiang in [Jia81] shows that with the exceptions of the 0 and 2-twist knots, the algebraically slice twist knots are linearly independent in the concordance group.

The 1-twist knot is of order 2 in the concordance group. Livingston and Naik in [LN99] show that infinitely many of those twist knots that are algebraically of order 4 are not of finite order in $\mathcal{C}$. Tamulis in [Tam02] finds an infinite linearly independent set of algebraic order 2 twist knots. Kim shows that no nontrivial linear combination of the twist knots (with the exception of the unknot, stevedore knot and figure eight knot) is ribbon [Kim05]. Results of Lisca show that the twist knots (with the same exceptions) are linearly independent in the *smooth* concordance group [Lis07].

This chapter is devoted to the proof that of the twist knots that are algebraically of order 2 almost are all linearly independent:

Theorem 6.0.5. *The set containing all of the twist knots T_n which are algebraically of order two is linearly independent in $\mathcal{C}/\mathcal{F}_{1.5}$ with 39 possible exceptions:*

$$\begin{aligned} n &= 1,3,4,9,10,11,15,16,18,22,24,25,27,28,29,34,36,37,38, \\ &\quad 39,45,48,49,51,55,58,61,64,66,67,69,70,78,79,83,84,87, \\ &\quad 93,101. \end{aligned}$$

We will begin the process of proving this theorem with an observation. The Alexander polynomial of T_n is given by $\Delta_{T_n}(t) = nt^2 - (2n+1)t - n$. This polynomial is prime provided that n is not of the form $n = a^2 + a$, for some $a \in \mathbb{Z}$. Thus, Theorem 3.4.2 applies to give us the following fact.

Theorem 6.0.6. *The set of twist knots $\mathcal{T} := \{T_n : n > 0, n$ is not of the form $a^2 + a,$ and $\rho^1(T_n) \neq 0\}$ is linearly independent in the concordance group.*

The next step in the analysis of the twist knots, then, is to use the techniques of Chapter 4 to approximate $\rho^1(T_n)$. With an eye to Corollary 4.1.5 consider the derivative of $T_k\#T_k$ shown in Figure 6.1 in the case that T_k is algebraically of order 2.

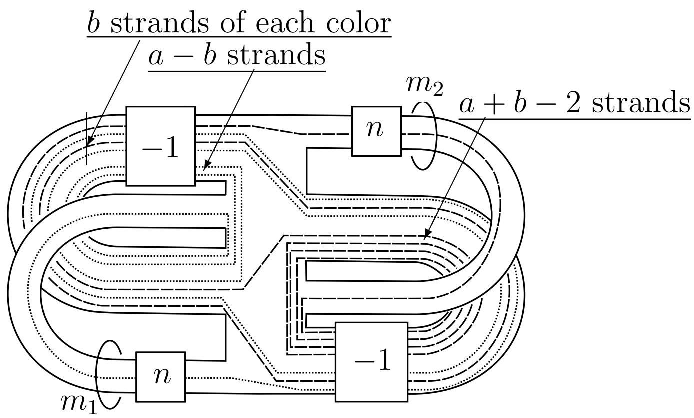

Figure 6.1: For $n = a^2 - a + b^2$, the link $L_{a,b}$ is a derivative for T_n The curves m_1 and m_2 are meridians about the bands on which the components of L sit.

Lemma 6.0.7. *The n-twist knot, T_n, is algebraically of order at most 2 if and only if there are positive integers a and b such that $n = a^2 - a + b^2$. For such an n, $L_{a,b}$, depicted in Figure 6.1, is a derivative for $T_n\#T_n$. Moreover, the meridians, m_1 and m_2, of the bands on which the components of $L_{a,b}$ sit are $\mathbb{Z}$-linearly independent of the components of $L_{a,b}$ in the Alexander module of $T_n\#T_n$.*

Proof. According to Levine [Lev69, Corollary 23], $T_n\#T_n$ is algebraically slice if and only if $4n + 1$ has no odd multiplicity prime factors congruent to 3 mod 4. Under these conditions, an elementary fact from number theory (see [Bur80, Theorem 12.3] for example) implies that there exist integers a_0 and b_0 with $a_0^2 + b_0^2 = 4n + 1$. Since $4n + 1$ is odd it must be that $a_0 = 2a - 1$ is odd while $b_0 = 2b$ is even. Under these conditions, this equation reduces to $a^2 - a + b^2 = n$ as was claimed.

We now prove that the link $L_{a,b}$ is a derivative. The first homology of F is a free abelian group with basis given by the curves $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ in Figure 6.2. With respect to this basis the components of $L_{a,b}$ represent the classes in $H_1(F)$ given by

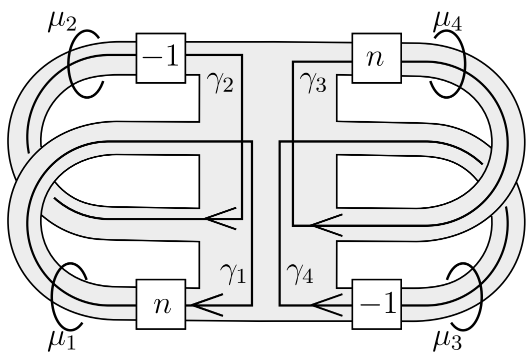

Figure 6.2: $\gamma_1, \ldots, \gamma_4$ form a basis for the first homology of a Seifert surface for $T_n \# T_n$. The meridians for γ_i, $\mu_1, \ldots, \mu_4$ form a generating set for the Alexander module of $T_n \# T_n$.

the vectors

$$l_1 = \begin{bmatrix} 1 & a & 0 & b \end{bmatrix}^T, l_2 = \begin{bmatrix} 0 & b & 1 & (1-a) \end{bmatrix}^T$$

while the Seifert form is given by

$$V = \begin{bmatrix} n & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & n & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

A computation (remembering that $n = a^2 - a + b^2$) verifies that $v_i^T V v_j = 0$ for $i, j \in \{1, 2\}$. Thus, $L_{a,b}$ is a derivative.

In order to address the linear independence claim, we first recall some classical knot theory facts [Rol90, Chapter 8, Section C]. The Alexander module, $A_0(T_n \# T_n)$, is the $\mathbb{Q}[t, t^{-1}]$-module generated by $\mu_1, \mu_2, \mu_3, \mu_4$ with presentation matrix $V - tV^T$. With respect to this presentation, $m_1 = \mu_1$ and $m_2 = \mu_3$ correspond to the first and third generators and γ_i corresponds to the i^{th} column of V.

Manipulating this presentation, we see that m_1 and m_2 generate the Alexander module and that

$$A_0(T_n \# T_n) \cong \left(\frac{\mathbb{Q}[t, t^{-1}]}{(nt^2 - (2n+1)t + n)} \right)^{\oplus 2}.$$

The components of $L_{a,b}$ (represented by l_1 and l_2) represent the module elements

$$\begin{aligned} l_1 &\mapsto (n + a(1 - n + nt))m_1 + b(1 - n + nt)m_2, \\ l_2 &\mapsto b(1 - n + nt)m_1 + (1 + (1 - a)(1 - n + nt))m_2. \end{aligned}$$

It is straightforward to see that images of m_1, m_2, v_1 and v_2 form a $\mathbb{Q}$-linearly independent set. This completes the proof.

□

6.1 Tools for computing the Cimasoni-Florens signature

In light of Theorems 4.1.4 and 5.4.3, to show that $\rho^1(T_n) = \frac{1}{2}\rho^1(T_n \# T_n)$ is not zero, it suffices to show that the integral of the Cimasoni-Florens signature of the link $L_{a,b}$ is greater than 1 in absolute value. In this section we develop the tools used to do so. All of the results of this section are given by integrating results proven in [CF08]. For an n colored link L, we let $R(L) = \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} \widehat{\sigma}_L(\omega) d\omega$ be the integral of the Cimasoni-Florens signature of L over $\mathbb{T}^n$ with respect to normalized Lebesgue measure.

Cimasoni and Florens [CF08, Proposition 5.1] show that if colored links L and L' differ by either of the the moves indicated in Figure 6.3 then for all ω, $\sigma_L(\omega)$ and $\sigma_{L'}(\omega)$ differ by 0 or 1 or -1 depending on the Conway potential function of L and L'. Integrating this result gives that $R(L)$ and $R(L')$ differ by at most 1.

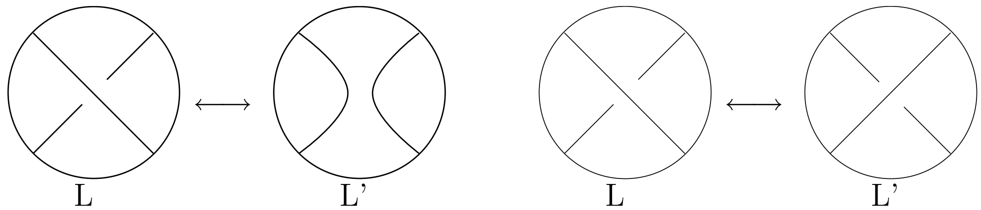

Figure 6.3: Moves which change the Cimasoni-Florens signature by at most one. The move on the left is understood to preserve orientations and be between arcs of the same color. The move of the right is understood to be between arcs of different colors

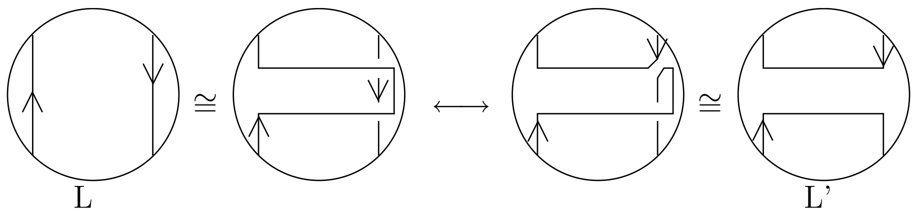

Figure 6.4: A band addition realized via a smoothing

The addition of a band between two arcs of the same color as in Figure 6.4 can be realized by a single smoothing. Thus, if L and L' differ by either a band addition or a crossing change between arcs of the same color, then $R(L)$ and $R(L')$ differ by at most 1. Additionally if this band runs between split sublinks of L, then $\sigma_L = \sigma_{L'}$ [CF08, Proposition 2.12]. Summarizing these results, we have the following facts.

1. If the colored links L and L' differ by a crossing change between arcs of different colors, then $|R(L) - R(L')| \leq 1$

2. If the colored links L and L' differ by a smoothing a crossing between arcs of the same color, then $|R(L) - R(L')| \leq 1$

3. If the links L and L' differ by the addition of a band between arcs of the same color, then $|R(L) - R(L')| \leq 1$. If this band joins split sublinks of L then $R(L) = R(L')$.

The local move on colored links depicted in Figure 6.5 will be relevant in our analysis of $L_{a,b}$. We provide bounds on the the effect it can have on the R-invariant.

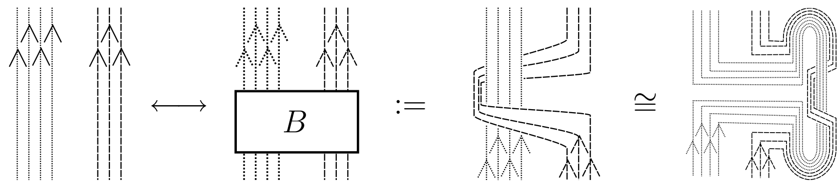

Figure 6.5: The move considered in Proposition 6.1.1 realized via band summing with the link V. The two different bands of arcs are assumed to be of different colors.

Proposition 6.1.1. *If the colored links L and L' differ by the move depicted in Figure 6.5 with a strands of one color and b strands of a different color then* $|R(L) - R(L')| \leq a + b - 1$

Proof. As is shown in Figure 6.5, L' is given by taking the split union of L with the two colored link V depicted in Figure 6.6 and adding $a + b$ bands. The first of these bands runs between split components of $L \sqcup V$. The signature invariant adds under split union [CF08, Proposition 2.12]. It follows that $|R(L') - R(L) - R(V)| \leq a+b-1$, where V is the link depicted in Figure 6.6.

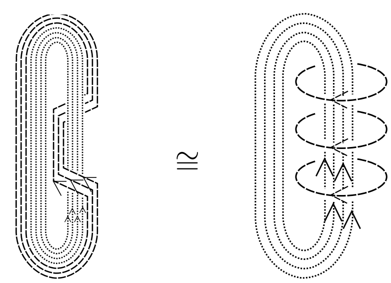

Figure 6.6: The link V and an isotopy.

In order to compute $R(V)$, notice that the result of reversing the orientation all of the components of V of one color is isotopic the mirror image of V. By [CF08, Proposition 2.10 and Corollary 2.11], $\sigma_V(\omega_1, \omega_2^{-1}) = -\sigma_V(\omega_1, \omega_2)$. Since $\omega \mapsto \omega^{-1}$ is a measure preserving transformation of $\mathbb{T}^1$, this implies that $R(V) = 0$ and completes the proof. $\square$

6.2 Performing the computation for $L_{a,b}$

In this section we perform the promised computation of $\rho^0(L_{a,b})$, which is equal to the integral of the Cimasoni-Florens signature of $L_{a,b}$. We begin by isotoping the $L_{a,b}$ to the link in Figure 6.7. Recall that we use the notaiton $R(L)$ to denote the integral of the Cimasoni-Florens signature of the colored link L, even when the conditions of Theorem 5.4.3 are not satisfied.

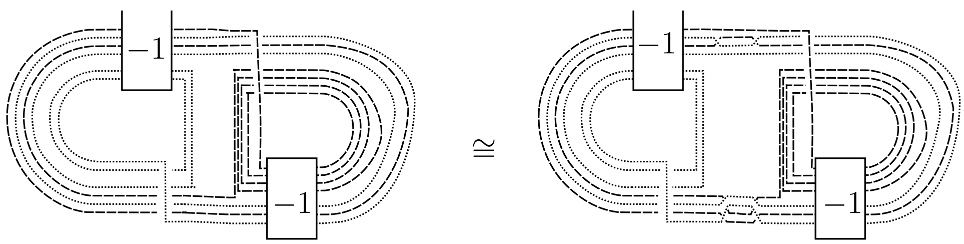

Figure 6.7: The link $L_{a,b}$ after an isotopy.

By adding $2b$ bands to this link, as in Figure 6.4, b of each color, the link $L_{a,b}$ is reduced to the split union of two links, $L^1_{a,b}$ depicted in Figure 6.8. Thus, $|\rho^0(L_{a,b}) - R(L^1_{a,b})| \leq 2b-1$.

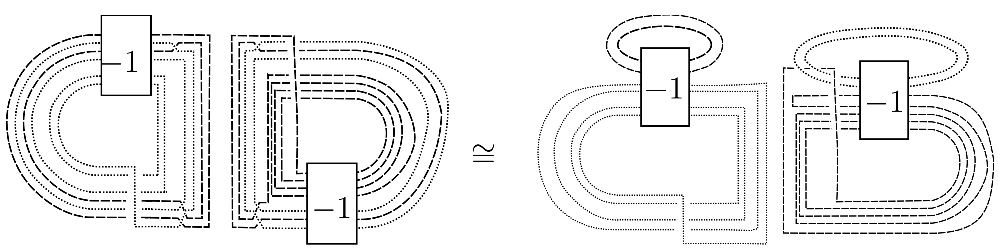

Figure 6.8: The link $L^1_{a,b}$ and an isotopy.

By changing b crossings between components of different colors, $L^1_{a,b}$ becomes the link $L^2_{a,b}$ of Figure 6.9 so that $|R(L^1_{a,b}) - R(L^2_{a,b})| \leq b$ and $|\rho^0(L_{a,b}) - R(L^2_{a,b})| \leq 3b-1$.

Figure 6.9 realizes $L^2_{a,b}$ as the result of performing the move depicted in Figure 6.5 twice. Recall that this move is denoted with a B. The left-most twist involves a strands of one color and b of the other. The right-most involves $a-1$ of one color and b of the other.

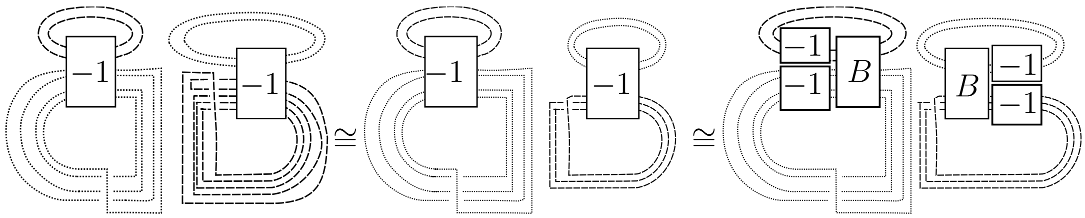

Figure 6.9: The link $L^2_{a,b}$ and some isotopies. B denotes the move in Figure 6.5.

Proposition 6.1.1 asserts that these moves will change the R-invariant by at most $(a+b-1)+(a-1+b-1)$. If we let $L^3_{a,b}$ be the link obtained by removing these twists from the diagram then

$$|R(L^2_{a,b}) - R(L^3_{a,b})| \leq (a+b-1)+(a-1+b-1).$$

By the triangle inequality,

$$|\rho^0(L_{a,b}) - R(L^3_{a,b})| \leq 5b+2a-4.$$

Figure 6.10 depicts this link. It is significant that $L^3_{a,b}$ is the split union of two one colored $(b,-b)$-torus links, an $(a,1-a)$-torus knot, and a $(1-a,a)$-torus knot.

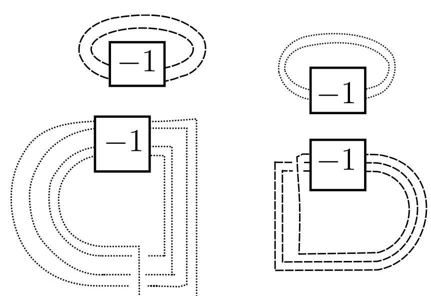

Figure 6.10: The link $L^3_{a,b}$ consists of the split union of two one colored $(b,-b)$-torus links, an $(a,1-a)$-torus knot and a $(1-a,a)$-torus knot.

The R-invariant adds under split union [CF08, Proposition 2.13].

$$R(L^3_{a,b}) = R(T(a,1-a)) + R(T(1-a,a)) + 2R(T(b,-b)),$$

where $T(a,b)$ is the one colored (a,b)-torus link. In the case of knots, the Cimasoni-Florens signature agrees with the Tristram-Levine signature. The integral of the Tristram-Levine signature of torus knots is computed by Borodzic [Bor] and independently by Collins [Col]. In [Bor], Borodzic also computes the integral of the one colored signature of the (b,b)-torus link, which is the mirror image of the $(b,-b)$-torus link. Specifying these results to our setting,

$$\begin{aligned} R(T(a,1-a)) &= R(T(1-a,a)) = \frac{(a+1)(a-2)}{3}, \\ R(T(b,-b)) &= \frac{(b-1)^2}{3} \end{aligned}$$

Putting all of this together,

$$\begin{aligned} \rho^0(L_{a,b}) &\geq \frac{2(a+1)(a-2)+2(b-1)^2}{3} - (2a+5b-4) \\ &= \frac{2a^2+2b^2-8a-19b+10}{3} \end{aligned} \tag{6.1}$$

Since we are interested in when $\rho^0(L_{a,b}) > 1$, we ask when

$$f(a,b) := 2a^2+2b^2-8a-19b+7 > 0$$

By observing that $f(a,b)$ grows quadratically in both a and b, we immediately see that with only finitely many exceptions, $\rho^1(T_n) > 0$.

We use a computer to find $\{n : n = a^2-a+b, a > 0, b > 0, f(a,b) > 0\}$ and combine this result with the computation in [Dav10, Corollary 6.2] showing that $\rho^0(L_{a,1}) > 1$ for $a \geq 3$. In doing so we find that

$$\begin{aligned} n = \; & 1,3,4,9,10,11,15,16,18,22,24,25,27,28,29,34,36,37,38, \\ & 39,45,48,49,51,55,58,61,64,66,67,69,70,78,79,83,84,87, \\ & 93,101. \end{aligned}$$

are all of the possible exceptions to the claim that if $\rho^1(T_n) \neq 0$. This completes the proof of Theorem 6.0.5.

$\square$

Chapter 7

Removing ambiguity from the [CHL10b] construction of linearly independent sets

In [CHL10b], Cochran-Harvey-Leidy provide a construction of infinite rank free abelian subgroups arbitrarily deep in the solvable filtration of the knot concordance group. Due to the difficulty of computing first order signatures, their construction is not explicit. In this section we use Theorems 4.1.4 and 5.4.3 to perform this computation.

7.1 Background: robust doubling operators

We recall the operation used in [CHL10b] to produce generating sets for infinite rank subgroups of $\mathcal{F}_n/\mathcal{F}_{n.5}$. A pair (R, η), also denoted R_η is called a doubling operator if R is a slice knot and η is an unknotted curve disjoint from R. For an example, see Figure 7.2. The justification for calling such an object a doubling operator is the process of infection, which takes a doubling operator R_η, and a knot K, and returns a new knot $R_\eta(K)$ given by taking the strands of R which pass through the disk bounded by η and tying them into the knot K. An illustration is provided in Figure 7.1

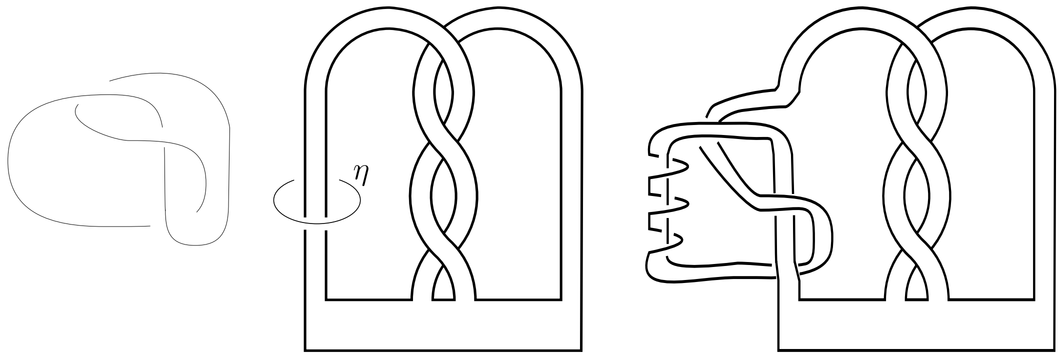

Figure 7.1: Left to right: (1) The trefoil knot, J. (2) A doubling operator R_η. (3) The result of infection, $R_\eta(J)$.

In [CHL10b], they study doubling operators satisfying some additional conditions

Definition (Definition 7.2 of [CHL10b]). For R a slice knot and α a curve in the complement of R with zero linking number with R, R_α is called a robust doubling operator if the following conditions hold.

1. The rational Alexander module of R is generated by α, and for a prime polynomial δ
$$A_0(R) \cong \frac{\mathbb{Q}[t^{\pm 1}]}{\langle \delta(t)\delta(t^{-1}) \rangle}.$$
2. For every isotropic submodule $P \subseteq A_0(R)$ either $\rho(M(R), \phi_P) \neq 0$ or $P = \ker(A_0(R) \to A_0(E))$ for E the complement in the 4-ball of a slice disk bounded by R.

Cochran-Harvey-Leidy show that iterating the infection procedure using robust doubling operators increases the solvability of the knots, that is, if K is $(n-1)$-solvable and R_η is robust then $R_\eta(K)$ is (n)-solvable. They go on to show that under some additional assumptions it does not increase solvability any further, even resulting in knots for which no nontrivial linear combination is $(n+1)$-solvable:

Theorem 7.1.1 (Theorem 7.5 [CHL10b]). *Let R_η be a Robust doubling operator. Let $\{K_j\}_{j=1}^n$ be a set of knots in $\mathcal{F}_0$ such that the set of abelian ρ-invariants $\{\rho_0(K_j)\}_{j=1}^n$ is rationally linearly independent of the first order signatures $\{\rho^1_P(M(R)) : P \subseteq A_0(R)$ is isotropic.$\}$.*

Then $\{(R_\eta)^n(K_j) = R_\eta(R_\eta(\dots R_\eta(K_j)\dots))\}_{j=1}^n$ is linearly independent in $\mathcal{F}_n/\mathcal{F}_{n.5}$.

In that paper, Cochran-Harvey-Leidy verify that the doubling operators (R_k, η_k) and (R'_k, η'_k) $k > 0$ in Figure 7.2 satisfy every condition of a robust doubling operator except that $\rho^1(R_k)$, the first order ρ-invariant corresponding to the trivial isotropic module, which cannot correspond to a slice disk, might vanish. In [CHL10b] they have no means of computing this ρ-invariant. If T is a knot with nonzero ρ^0-invariant, then $\rho^1(R) \neq \rho^1(R')$. At least one is nonzero. Thus, they get at least one robust doubling operator for each positive integer k but they cannot verify that any one of these doubling operators is robust.

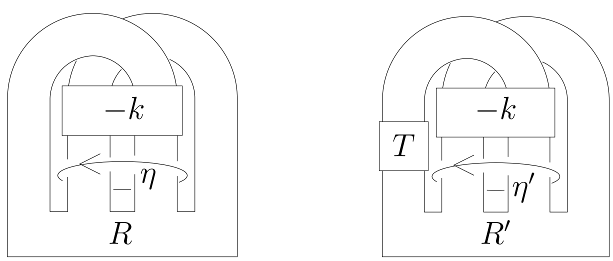

Figure 7.2: The doubling operators of [CHL10b] at least one of which is robust. The $(-k)$ denotes k full negative twists between the bands. T is a knot with nonzero ρ^0-invariant. We verify that R_η is robust for $k \geq 3$.

7.2 Explicit robust doubling operators

We prove the following theorem, and thus remove this ambiguity from the construction of [CHL10b].

Theorem 7.2.1. *For $k \geq 3$, $\rho^1(R_k) \neq 0$ and (R_k, η_k) is a robust doubling operator.*

By definition, the ρ^1-invariant is the first order signature corresponding to the trivial submodule of the Alexander module. We outline a strategy for its computation. By Theorem 3.2.7, if Q is a submodule of $A_0(R_K\#R_k) = A_0(R_k) \oplus A_0(R_k)$ such that $Q \cap (A_0(R_k) \times \{0\})$ and $Q \cap \{0\} \times A_0(R_k))$ are trivial, then

$$\rho^1_Q(R_k\#R_k) = \rho^1_{Q\cap(A_0(R_k)\times\{0\})}(R_k) + \rho^1_{Q\cap(\{0\}\times A_0(R_k))}(R_k) = 2\rho^1(R_k).$$

If we can find a submodule $Q \subseteq A_0(R_k\#R_k)$ which intersects the two $A_0(R_k)$ direct summands trivially and we can compute $\rho^1_Q(R_k\#R_k)$ then we will know $\rho^1(R_k)$. Theorem 4.1.4 is the tool we will use to compute $\rho^1_Q(R_k\#R_k)$. We must also find a derivative which generates this submodule, Q. The following proposition gives a derivative which generates such a submodule.

Proposition 7.2.2. *The link L_k depicted in Figure 7.3 is a derivative for $R_k\#R_k$. The set gotten by lifting the meridians m_1, m_2 of the bands on which the components of L sit and the components of L to the Alexander module is $\mathbb{Q}$-linearly independent. The submodule generated by L_k in $A_0(R_k\#R_k) = A_0(R_k) \oplus A_0(R_k)$ intersects the $A_0(R_k)$-summands trivially.*

Thus, $|2\rho^1(R_k) - \rho^0(L_k)| \leq 1$.

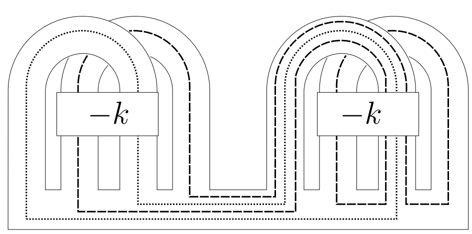

Figure 7.3: The Link, L_k, depicted above is a derivative of $R_k\#R_k$.

Proof. With respect to the standard basis for the first homology of the obvious Seifert surface bounded by $R_k \# R_k$, the Seifert matrix is given by

$$V = \begin{bmatrix} 0 & k & 0 & 0 \\ k+1 & 0 & 0 & 0 \\ 0 & 0 & 0 & k \\ 0 & 0 & k+1 & 0 \end{bmatrix}$$

while the components of L_k are given by

$$l_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, l_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}.$$

Similar to the analysis in the proof of Lemma 6.0.7, we see that

$$A_0(R_k \# R_k) \cong \frac{\mathbb{Q}[t,t^{-1}]}{kt-(k+1)} \oplus \frac{\mathbb{Q}[t,t^{-1}]}{(k+1)t-k} \oplus \frac{\mathbb{Q}[t,t^{-1}]}{kt-(k+1)} \oplus \frac{\mathbb{Q}[t,t^{-1}]}{(k+1)t-k}.$$

The components of L and the meridians about the bands on which L sit, m_1, m_2 lift to

$$l_1 \mapsto \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, l_2 \mapsto \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, m_1 \mapsto \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, m_2 \mapsto \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

which clearly form a $\mathbb{Q}$-linearly independent set.

Finally the $A_0(R_k)$-direct summands coming from the R_k-connected summands are spanned by

$$A_0(R_k) \oplus \{0\} = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right), \{0\} \oplus A_0(R_k) = \text{span}\left(\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right).$$

Clearly, the span of l_1 and l_2 intersect each of these summands trivially. $\square$

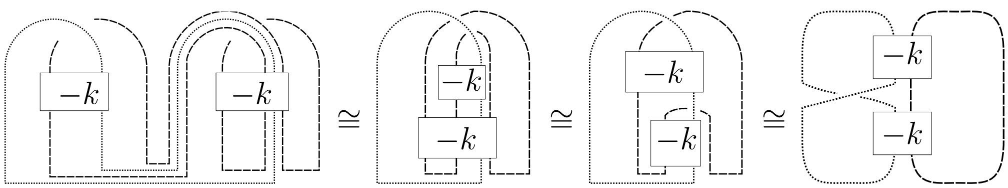

Figure 7.4: The Link, L_k, depicted above with some isotopies, is a derivative of $R_k \# R_k$.

Proof of Theorem 7.2.1. To prove that (R_k, η_k) is robust it suffices to show that $|\rho^0(L_k)| > 1$. Consider the sequence of isotopies in Figure 7.4. After the final isotopy, it is clear that the link L_k bounds a C-complex, S consisting of two disks with $2k$ clasps between them. The collection of loops $\{\alpha_1, \dots \alpha_{k-1}, \beta_1, \dots, \beta_{k-1}, \gamma\}$ depicted in Figure 7.5 form a basis for $H_1(S)$. We compute the linking numbers between these loops and their pushoffs.

$$\operatorname{lnk}(\alpha_j, \alpha_{j+1}^{\epsilon_1,\epsilon_2}) = \begin{cases} -1 & \text{if } \epsilon_1 = \epsilon_2 = 1, \\ 0 & \text{otherwise.} \end{cases}$$

$$\operatorname{lnk}(\alpha_j, \alpha_{j-1}^{\epsilon_1,\epsilon_2}) = \operatorname{lnk}(\alpha_{j-1}, \alpha_j^{-\epsilon_1,-\epsilon_2}) = \begin{cases} -1 & \text{if } \epsilon_1 = \epsilon_2 = -1, \\ 0 & \text{otherwise.} \end{cases}$$

$$\operatorname{lnk}(\beta_i, \alpha_j^{\epsilon_1,\epsilon_2}) = \operatorname{lnk}(\alpha_i, \beta_j^{\epsilon_1,\epsilon_2}) = 0 \text{ for all } i, j$$

$$\operatorname{lnk}(\beta_i, \beta_j^{\epsilon_1,\epsilon_2}) = \operatorname{lnk}(\alpha_i, \alpha_j^{\epsilon_1,-\epsilon_2})$$

$$\operatorname{lnk}(x, \gamma^{\epsilon_1,\epsilon_2}) = \begin{cases} -1 & \text{if } x = \alpha_k \text{ and } \epsilon_1 = \epsilon_2 = 1 \\ +1 & \text{if } x = \beta_k, \epsilon_1 = 1 \text{ and } \epsilon_2 = -1 \\ +1 & \text{if } x = \gamma \\ 0 & \text{otherwise} \end{cases}$$

$$\operatorname{lnk}(\gamma, x^{\epsilon_1,\epsilon_2}) = \begin{cases} -1 & \text{if } x = \alpha_k \text{ and } \epsilon_1 = \epsilon_2 = -1 \\ +1 & \text{if } x = \beta_k, \epsilon_1 = -1 \text{ and } \epsilon_2 = 1 \\ +1 & \text{if } x = \gamma \\ 0 & \text{otherwise.} \end{cases}$$

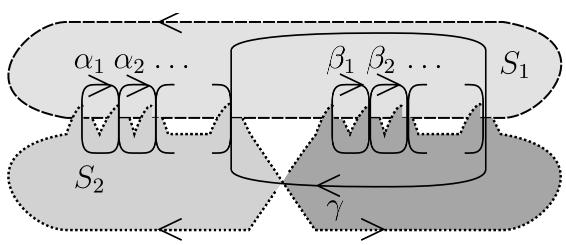

Figure 7.5: A C-complex $S = S_1 \cup S_2$ for the link L_k. The loops $\alpha_1, \ldots, \alpha_{k-1}$, $\beta_1, \ldots, \beta_{k-1}$, γ give a basis for its first homology.

Taking this information and recalling that $H(\omega_1, \omega_2) = \sum_{\epsilon \in \{\pm 1\}^2} (1-\omega_1^{\epsilon_1})(1-\omega_2^{\epsilon_2}) A^{\epsilon_1,\epsilon_2}$

provides the $2k-1 \times 2k-1$ Cimasoni-Florens linking matrix for L_k:

$$H(\omega_1, \omega_2) =$$
$$\begin{bmatrix} G(1-\omega_1, 1-\omega_2) & 0^{k-1\times k-2} & -\nu(1-\omega_1, 1-\omega_2) \\ 0^{k-1\times k-2} & G(1-\omega_1, 1-\omega_2^{-1}) & \nu(1-\omega_1, 1-\omega_2^{-1}) \\ -\nu(1-\omega_1^{-1}, 1-\omega_2^{-1})^T & \nu(1-\omega_1^{-1}, 1-\omega_2)^T & 4\mathfrak{Re}(1-\omega_1)\mathfrak{Re}(1-\omega_2) \end{bmatrix}$$

where

$$G(z_1, z_2) = \begin{bmatrix} 2\mathfrak{Re}(z_1\overline{z_2}) & -z_1\overline{z_2} & & 0 \\ -\overline{z_1}z_2 & 2\mathfrak{Re}(z_1\overline{z_2}) & \ddots & \\ & \ddots & \ddots & -z_1\overline{z_2} \\ 0 & & -\overline{z_1}z_2 & 2\mathfrak{Re}(z_1\overline{z_2}) \end{bmatrix}, \nu(z_1, z_2) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ z_1 z_2 \end{bmatrix}$$

and $\mathfrak{Re}(z) = \frac{z+\bar{z}}{2}$ is the real part of the complex number z. Notice that $H(\omega_1, \omega_2)$, while not quite tridiagonal, consists of the direct sum of two tridiagonal matrices together with an additional row and column. Diagonalizing G gives

$$H(\omega_1, \omega_2) \sim$$
$$\begin{bmatrix} D(1-\omega_1, 1-\omega_2) & 0 & -\nu(1-\omega_1, 1-\omega_2) \\ 0 & D(1-\omega_1, 1-\omega_2^{-1}) & \nu(1-\omega_1, 1-\omega_2^{-1}) \\ -\nu(1-\omega_1^{-1}, 1-\omega_2^{-1})^T & \nu(1-\omega_1^{-1}, 1-\omega_2)^T & 4\mathfrak{Re}(1-\omega_1)\mathfrak{Re}(1-\omega_2) \end{bmatrix}$$

where $D(z_1, z_2)$ is the diagonal matrix with entries $p_1(z_1z_2), \ldots p_{k-1}(z_1z_2)$ satisfying the recurrence relation

$$p_1(z) = 2\mathfrak{Re}(z), p_{j+1}(z) = 2\mathfrak{Re}(z) - \frac{|z|^2}{p_j(z)}. \tag{7.1}$$

This recurrence relation is solved by $p_j(z) = \frac{\mathfrak{Im}(z^{j+1})}{\mathfrak{Im}(z^j)}$ where $\mathfrak{Im}(z)$ denotes the imaginary part of z. Making this substitution, and completing the diagonalization of H gives that H is similar to a diagonal matrix with entries

$$
\begin{gathered}
q_1(z_1,z_2) := \frac{\mathfrak{Im}\big((z_1z_2)^2\big)}{\mathfrak{Im}((z_1z_2))}, \quad \ldots \quad q_{k-1}(z_1,z_2) := \frac{\mathfrak{Im}\big((z_1z_2)^k\big)}{\mathfrak{Im}\big((z_1z_2)^{k-1}\big)}, \\
q_k(z_1,z_2) := \frac{\mathfrak{Im}\big((z_1\overline{z_2})^2\big)}{\mathfrak{Im}((z_1\overline{z_2}))}, \quad \ldots \quad q_{2k-2}(z_1,z_2) := \frac{\big(\mathfrak{Im}(z_1\overline{z_2})^k\big)}{\mathfrak{Im}\big((z_1\overline{z_2})^{k-1}\big)}, \\
q_{2k-1}(z_1,z_2) := 4\mathfrak{Re}(z_1)\mathfrak{Re}(z_2) - \frac{|z_1z_2|^2\mathfrak{Im}\big((z_1z_2)^{k-1}\big)}{\mathfrak{Im}\big((z_1z_2)^k\big)} - \frac{|z_1z_2|^2\mathfrak{Im}\big((z_1\overline{z_2})^{k-1}\big)}{\mathfrak{Im}\big((z_1\overline{z_2})^k\big)}.
\end{gathered}
$$

Thus, $\sigma_L(\omega_1,\omega_2) = \sigma(H)$ is given by the sum of the signs of these diagonal terms and $\rho^0(L_k)$ is given by the sum of the integrals of their signs, that is,

$$
\rho^0(L_k) = \sum_{j=1}^{2k-1} \frac{1}{(2\pi)^2} \int\int_{\mathbb{T}^2} \operatorname{sign}(q_j((1-\omega_1)(1-\omega_2)))d\omega_1 d\omega_2. \tag{7.2}
$$

We start with the first $k-1$ terms in this sum. Recall that for $\omega = \omega_1$ or ω_2, $z = 1-\omega = 1-e^{i\theta} = \sqrt{2-2\cos(\theta)}e^{-i\theta/2}$ for $-\pi < \theta \le \pi$. Making this substitution,

$$
\begin{aligned}
q_j(z_1,z_2) &= \frac{\mathfrak{Im}(z_1z_2)^{j+1}}{\mathfrak{Im}(z_1z_2)^j} \\
&= \frac{\mathfrak{Im}(|z_1z_2|^{j+1}e^{i(j+1)(\theta_1+\theta_2)/2})}{\mathfrak{Im}(|z_1z_2|^j e^{ij(\theta_1+\theta_2)/2})} \\
&= |z_1z_2|\frac{\sin((j+1)(\theta_1+\theta_2)/2)}{\sin(j(\theta_1+\theta_2)/2)}
\end{aligned}
\tag{7.3}
$$

which has the sign as

$$
\frac{\sin((j+1)(\theta_1+\theta_2)/2)}{\sin(j(\theta_1+\theta_2)/2)}.
$$

Thus,

$$
\tfrac{1}{(2\pi)^2}\int\int_{\mathbb{T}^2} \operatorname{sign}(q_j((1-\omega_1)(1-\omega_2)))d\omega_1 d\omega_2 = \tfrac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} F\left(\tfrac{\theta_1+\theta_2}{2}\right)d\theta_1 d\theta_2,
$$

where $F(\theta) = \operatorname{sign}\left(\frac{\sin((j+1)\theta)}{\sin(j\theta)}\right)$. The integrand and domain are symmetric about the line $\theta_1 = -\theta_2$, so that

$$\frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} F\left(\frac{\theta_1+\theta_2}{2}\right) d\theta_1 d\theta_2 = \frac{2}{(2\pi)^2}\int_{\theta_1=-\pi}^{\pi}\int_{\theta_2=-\theta_1}^{\pi} F\left(\frac{\theta_1+\theta_2}{2}\right) d\theta_1 d\theta_2.$$

Making the substitution $u = \frac{\theta_1+\theta_2}{2}$, $v = \frac{\theta_1-\theta_2}{2}$, transforms this integral to

$$\begin{aligned}
\tfrac{1}{(2\pi)^2}\int_{\theta_1=-\pi}^{\pi}\int_{\theta_2=-\theta_1}^{\pi} F\left(\frac{\theta_1+\theta_2}{2}\right) d\theta_1 d\theta_2 &= \tfrac{2}{(2\pi)^2}\int_{u=0}^{\pi}\int_{v=u-\pi}^{\pi-u} F(u)2dvdu \\
&= \tfrac{2}{(2\pi)^2}\int_{u=0}^{\pi} 4(\pi-u)F(u)du \\
&= \tfrac{2}{\pi^2}\int_{u=0}^{\pi} (\pi-u)F(u)du
\end{aligned}$$

Notice that $F(u) = -F(\pi-u)$. Taking $v = \pi - u$ transforms this into

$$\tfrac{1}{(2\pi)^2}\int_{\theta_1=0}^{2\pi}\int_{\theta_2=0}^{2\pi-\theta_1} F\left(\frac{\theta_1+\theta_2}{2}\right) d\theta_1 d\theta_2 = \frac{2}{\pi^2}\int_{u=0}^{\pi} uF(u)du$$

Finally, since $F(u) = +1$ for sufficiently small positive values of u and $F(u)$ changes sign every multiple of $\frac{\pi}{j}$ and of $\frac{\pi}{j+1}$, $F(u) = +1$ on $\left(0, \frac{\pi}{j+1}\right) \cup \left(\frac{\pi}{j}, \frac{2\pi}{j+1}\right) \cup \cdots \cup \left(\frac{(j-1)\pi}{j}, \frac{j\pi}{j+1}\right)$ and $F(u) = -1$ on $\left(\frac{\pi}{j+1}, \frac{\pi}{j}\right) \cup \left(\frac{2\pi}{j+1}, \frac{2\pi}{j}\right) \cup \cdots \cup \left(\frac{j\pi}{j+1}, 1\right)$. Thus, we rewrite the integral as

$$\begin{aligned}
\tfrac{2}{\pi^2}\int_{u=0}^{\pi} uF(u)du &= \tfrac{2}{\pi^2}\left(\sum_{m=1}^{j}\int_{u=\frac{(m-1)\pi}{j}}^{\frac{m\pi}{j+1}} udu - \sum_{m=1}^{j}\int_{u=\frac{m\pi}{j+1}}^{\frac{m\pi}{j}} udu\right) \\
&= \tfrac{2}{\pi^2}\sum_{m=1}^{j}\left(\left(u^2/2\right)\Big|_{u=\frac{(m-1)\pi}{j}}^{\frac{m\pi}{j+1}} - \left(u^2/2\right)\Big|_{u=\frac{m\pi}{j+1}}^{\frac{m\pi}{j}}\right) \\
&= \sum_{m=1}^{j}\left(\frac{2m^2}{(j+1)^2} - \frac{(m-1)^2}{j^2} - \frac{m^2}{j^2}\right).
\end{aligned}$$

Finally we use the identity $\displaystyle\sum_{m=1}^{k} m^2 = \frac{(k)(k+1)(2k+1)}{6}$ to evaluate.

$$\frac{2}{\pi^2}\int_{u=0}^{\pi} uF(u)du = -\left(\frac{j^2+j+1}{3(j^2+j)}\right).$$

Since for $1 \leq j \leq 2k-1$, $q_{2k-1+j}(\omega_1,\omega_2) = q_j(\omega_1,\omega_2^{-1})$ and $\omega \mapsto \omega^{-1}$ is a measure preserving homeomorphism of $\mathbb{T}^1$, it follows that

$$\int_{\theta_1=0}^{2\pi}\int_{\theta_2=0}^{2\pi-\theta_1} \operatorname{sign}(q_j(\theta_1,\theta_2))d\theta_1 d\theta_2 \quad = \quad \int_{\theta_1=0}^{2\pi}\int_{\theta_2=0}^{2\pi-\theta_1} \operatorname{sign}(q_{2k-1+j}(\theta_1,\theta_2))d\theta_1 d\theta_2.$$

Equation (7.2) then becomes

$$\begin{aligned} \rho^0(L_k) &= -2\sum_{j=1}^{k-1}\frac{j^2+j+1}{3(j^2+j)} + \frac{1}{(2\pi)^2}\int\int_{\mathbb{T}^2} \operatorname{sign}(q_{2k-1}((1-\omega_1)(1-\omega_2)))d\omega_1 d\omega_2. \\ &= \frac{2-2k^2}{3k} + \frac{1}{(2\pi)^2}\int\int_{\mathbb{T}^2} \operatorname{sign}(q_{2k-1}((1-\omega_1)(1-\omega_2)))d\omega_1 d\omega_2. \end{aligned} \tag{7.4}$$

Notice that

$$E_k := \frac{1}{(2\pi)^2}\int\int_{\mathbb{T}^2} \operatorname{sign}(q_{2k-1}((1-\omega_1)(1-\omega_2)))d\omega_1 d\omega_2$$

has absolute value at most one, while, $\frac{2-2k^2}{3k} < -2$ for $k \geq 4$, so that without analyzing E_k any further, $\rho^0(L_k) < -1$ and R_k is robust. We next analyze the term E_k for $k = 1, 2, 3$.

For $k = 2$, $\frac{2-2k^2}{3k} = -1$. For $k = 3$, $\frac{2-2k^2}{3k} = -16/9$. Thus, to use these techniques to show that $\rho^0(L_k) < -1$ and conclude that (R_2,η_2) and (R_2,η_2) are robust, we must show that $E_2 < 0$ and $E_3 < 7/9$. Recall that

$$q_{2k-1}(z_1,z_2) = 4\mathfrak{Re}(z_1)\mathfrak{Re}(z_2) - \frac{|z_1z_2|^2\mathfrak{Im}\big((z_1z_2)^{k-1}\big)}{\mathfrak{Im}\big((z_1z_2)^k\big)} - \frac{|z_1z_2|^2\mathfrak{Im}\big((z_1\overline{z_2})^{k-1}\big)}{\mathfrak{Im}\big((z_1\overline{z_2})^k\big)}.$$

For $z = 1-e^{i\theta}$, $\mathfrak{Re}(z_j) = 1-\cos(\theta)$ and $|z| = \sqrt{2-2\cos(\theta)}$. Recall from equation (7.3) that $\frac{\mathfrak{Im}\left((z_1z_2)^{k-1}\right)}{\mathfrak{Im}\left((z_1z_2)^{k}\right)} = \frac{1}{|z_1z_2|}\frac{\sin((k-1)(\theta_1+\theta_2)/2)}{\sin(k(\theta_1+\theta_2)/2)}$. Making these substitutions:

$$\begin{gathered} q_{2k-1}(z_1, z_2) = \\ 4(1-\cos(\theta_1))(1-\cos(\theta_2)) - \\ 2\sqrt{(1-\cos(\theta_1))(1-\cos(\theta_2))}\left(\frac{\sin((k-1)(\theta_1+\theta_2)/2)}{\sin(k(\theta_1+\theta_2)/2)} + \frac{\sin((k-1)(\theta_1-\theta_2)/2)}{\sin(k(\theta_1-\theta_2)/2)}\right). \end{gathered}$$

Using the identity $1-\cos(\theta) = 2\sin(\theta/2)^2$, it follows that $q_{2k-1}(1-e^{i\theta_1}, 1-e^{i\theta_2})$ has the same sign for almost all (θ_1, θ_2) as

$$q'_{2k-1}(\theta_1, \theta_2) := \left(s_k^+ s_k^-\right)\left(4\left|s_1^+ s_1^-\right| s_k^+ s_k^- - s_{k-1}^+ s_k^- - s_k^+ s_{k-1}^-\right),$$

where $s_j^\pm = \sin(j(\theta_1 \pm \theta_2)/2)$.

The reason to make such a substitution is to replace q_{2k+1} with a function whose derivatives are uniformly bounded so that the integral of its sign is easier to approximate numerically. The bound coming from the product rule for differentiation and that

$$\left|\frac{\partial s_j^\pm}{\partial \theta_i}\right| = \frac{j}{2}\left|\cos\left(\frac{j(\theta_1 \pm \theta_2}{2}\right)\right| \leq \frac{j}{2}$$

gives $\left|\frac{\partial q'_{2k-1}}{\partial \theta_j}\right| \leq 12k+3$.

Thus if $q'(a,b) > (12k+3)\delta$ for some δ then $q' > 0$ on $[a-\delta/2, a+\delta/2] \times [b-\delta/2, b+\delta/2]$. An analogous statement about q' being negative also holds. Thus, if $S = \{S_1, \ldots S_N\}$ is a partition of $[-\pi,\pi] \times [-\pi,\pi]$ into squares each of length δ and

(x_i, y_i) is the center of S_i for $i = 1, \ldots N$, then

$$\begin{aligned}\sum_{i=1}^{N}(\mathrm{sign}(q'(x_i,y_i) - (12k-3)\delta))\delta^2 \\ \leq \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} q'(\theta_1,\theta_2)d\theta_1 d\theta_2 \\ \leq \sum_{i=1}^{N}(\mathrm{sign}(q'(x_i,y_i) + (12k-3)\delta))\delta^2\end{aligned}$$

Doing so suggests that E_2 is close to 0 and that E_3 is close to $\frac{1}{(2\pi^2)}(-6.36)$ which is about $-.16$. The bounds given above reveal that $-.283 \leq E_3 \leq -.008$. In particular, $E_3 < 7/9$, $\rho^0(L_3) < -1$, so that $\rho^1(R_3) < 0$ and (R_3, η_3) is robust.

The conclusion that E_2 is (seemingly) not negative means that our methods will not show that (R_2, η_2) is a robust doubling operator. $\square$

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